Faculty of Science, Technology, Engineering and Mathematics
M208 Pure Mathematics

Additional exercises for Book E

Contents

 WEB 07700 7
2.1

Additional exercises for Unit E1

Section 2

Additional Exercise E1

Determine whether each of the following is a subgroup of GL(2).

(a) $R = \left\{ \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} : a, c \in \mathbb{R},\ a \neq 0 \right\}$

(b) $S = \left\{ \begin{pmatrix} a & 0 \\ 1 & 1/a \end{pmatrix} : a \in \mathbb{R}^* \right\}$

(c) $T = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}$

(d) $V = \left\{ \begin{pmatrix} 1+n & n \\ -n & 1-n \end{pmatrix} : n \in \mathbb{Z} \right\}$

Section 3

Additional Exercise E2 Challenging

Show that the group $([0,1), +_1)$ (see Exercise E2 in Unit E1) is isomorphic to the group $(S^+(\bigcirc), \circ)$ of direct symmetries of the disc.

Section 4

Additional Exercise E3

Partition the symmetric group S_3 into left cosets of the subgroup $H = \langle (1\ 3) \rangle = \{e,\ (1\ 3)\}$.

Additional Exercise E4

The following table is the group table of a group $(G, \circ)$.

$\circ$	e	i	j	k	w	x	y	z
e	e	i	j	k	w	x	y	z
i	i	w	k	y	x	e	z	j
j	j	z	w	i	y	k	e	x
k	k	j	x	w	z	y	i	e
w	w	x	y	z	e	i	j	k
x	x	e	z	j	i	w	k	y
y	y	k	e	x	j	z	w	i
z	z	y	i	e	k	j	x	w

For each of the following subsets H of G, show that H is a subgroup of G and partition G into left cosets of H.

(a) $H = \{e, w\}$ (b) $H = \{e, i, w, x\}$

Additional Exercise E5

The following subgroup H of the symmetric group S_4 comprises all the elements that fix the symbol 1:

$$H = \{e,\ (2\ 3),\ (2\ 4),\ (3\ 4),\ (2\ 3\ 4),\ (2\ 4\ 3)\}.$$

(a) Without finding any elements of the left coset $(1\ 2)H$, show that each element of $(1\ 2)H$ maps the symbol 1 to the symbol 2.

(b) Hence write down the left coset $(1\ 2)H$. (You should be able to write down its elements without calculating any composites.)

(c) Using reasoning similar to that used in part (a), write down the other two left cosets of H in S_4 to complete the partition of S_4 into left cosets of H.

Additional Exercise E6

Partition the group $\mathbb{Z}_{18}$ into cosets of each of the following subgroups H.

(a) $H = \langle 9 \rangle = \{0, 9\}$ (b) $H = \langle 6 \rangle = \{0, 6, 12\}$

Additional Exercise E7

Partition the group $\mathbb{Z}_{13}^*$ into cosets of the subgroup $H = \langle 12 \rangle = \{1, 12\}$.

Additional Exercise E8

Partition the group $(\mathbb{Z}, +)$ into cosets of the subgroup $6\mathbb{Z} = \langle 6 \rangle$.

Additional Exercise E9

Partition the group $(G, \circ)$ whose group table is given in Additional Exercise E4 into *right* cosets of each of the following subgroups H.

(a) $H = \{e, w\}$ (b) $H = \{e, i, w, x\}$

(These are the same subgroups as in Additional Exercise E4.)

Additional Exercise E10

Partition the group S_4 into right cosets of the subgroup

$$H = \{e, (2\ 3), (2\ 4), (3\ 4), (2\ 3\ 4), (2\ 4\ 3)\}.$$

(This is the same subgroup as in Additional Exercise E5.)

Section 5

Additional Exercise E11

Let $(G, \circ)$ be the group whose group table was given in Additional Exercise E4. You saw there that each of the following sets H is a subgroup of $(G, \circ)$. Use your answers to Additional Exercises E4 and E9 to determine whether each of them is a normal subgroup of $(G, \circ)$.

(a) $H = \{e, w\}$ (b) $H = \{e, i, w, x\}$

Additional Exercise E12

Use your answers to Additional Exercises E5 and E10 to determine whether the subgroup

$$H = \{e, (2\ 3), (2\ 4), (3\ 4), (2\ 3\ 4), (2\ 4\ 3)\}$$

of S_4 is a normal subgroup.

Additional Exercise E13

In each of the following cases, give a reason why the subgroup H is normal in the group G.

(a) $G = \mathbb{Z}_{16}$, $H = \langle 4 \rangle = \{0, 4, 8, 12\}$.

(b) $G = S_5$, $H = A_5$.

Additional Exercise E14

Show that $H = \{e, k, w, z\}$ is a normal subgroup of the group $(G, \circ)$ with the following group table.

$\circ$	e	i	j	k	w	x	y	z
e	e	i	j	k	w	x	y	z
i	i	w	k	y	x	e	z	j
j	j	z	w	i	y	k	e	x
k	k	j	x	w	z	y	i	e
w	w	x	y	z	e	i	j	k
x	x	e	z	j	i	w	k	y
y	y	k	e	x	j	z	w	i
z	z	y	i	e	k	j	x	w

(This is the same group $(G, \circ)$ as in Additional Exercise E4.)

Additional Exercise E15

The following table is the group table of a group G.

	e	a	b	c	w	x	y	z
e	e	a	b	c	w	x	y	z
a	a	b	c	e	x	y	z	w
b	b	c	e	a	y	z	w	x
c	c	e	a	b	z	w	x	y
w	w	z	y	x	b	a	e	c
x	x	w	z	y	c	b	a	e
y	y	x	w	z	e	c	b	a
z	z	y	x	w	a	e	c	b

Show that each of the following sets is a normal subgroup of G.

(a) $\{e, b\}$ (b) $\{e, w, b, y\}$

Solutions to additional exercises for Unit E1

Solution to Additional Exercise E1

(a) The set R is a subset of the group GL(2), because each matrix

$$\begin{pmatrix} a & 0 \\ c & a \end{pmatrix} \quad (a \neq 0)$$

in R has determinant

$$a \times a - 0 \times c = a^2 \neq 0$$

and is therefore invertible. We show that the three subgroup properties hold for R.

SG1 Let $\mathbf{A}, \mathbf{B} \in R$. Then

$$\mathbf{A} = \begin{pmatrix} r & 0 \\ s & r \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} v & 0 \\ w & v \end{pmatrix},$$

for some $r, s, v, w \in \mathbb{R}$ with $r \neq 0$ and $v \neq 0$. So

$$\mathbf{AB} = \begin{pmatrix} r & 0 \\ s & r \end{pmatrix} \begin{pmatrix} v & 0 \\ w & v \end{pmatrix} = \begin{pmatrix} rv & 0 \\ sv + rw & rv \end{pmatrix}.$$

This matrix is of the form

$$\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$$

with $a = rv$ and $c = sv + rw$. Also $rv \neq 0$ since $r \neq 0$ and $v \neq 0$. Hence $\mathbf{AB} \in R$. Thus R is closed under matrix multiplication.

SG2 The identity element

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

of GL(2) is of the form

$$\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$$

with $a = 1$ and $c = 0$. So $\mathbf{I} \in R$.

SG3 Let $\mathbf{A} \in R$. Then

$$\mathbf{A} = \begin{pmatrix} r & 0 \\ s & r \end{pmatrix},$$

for some $r, s \in \mathbb{R}$ with $r \neq 0$. The inverse of $\mathbf{A}$ in GL(2) is

$$\mathbf{A}^{-1} = \frac{1}{r^2} \begin{pmatrix} r & 0 \\ -s & r \end{pmatrix} = \begin{pmatrix} 1/r & 0 \\ -s/r^2 & 1/r \end{pmatrix}.$$

This matrix is of the form

$$\begin{pmatrix} a & 0 \\ c & a \end{pmatrix}$$

with $a = 1/r$ and $c = -s/r^2$. Also $1/r \neq 0$. Hence $\mathbf{A}^{-1} \in R$. Thus R contains the inverse of each of its elements.

Since the three subgroup properties hold, R is a subgroup of GL(2).

(b) Subgroup property SG2 fails for S, because the identity element $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ of GL(2) is not in S, since its bottom left entry is not 1.

(c) The set T is a subset of the group GL(2), because each matrix

$$\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \quad (a \neq 0)$$

in T has determinant

$$a \times 1 - b \times 0 = a \neq 0$$

and is therefore invertible. We show that the three subgroup properties hold for T.

SG1 Let $\mathbf{A}, \mathbf{B} \in T$. Then

$$\mathbf{A} = \begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix},$$

for some $r, s, t, u \in \mathbb{R}$ with $r \neq 0$ and $t \neq 0$. So

$$\mathbf{AB} = \begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} rt & ru + s \\ 0 & 1 \end{pmatrix}.$$

This matrix is of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a = rt$ and $b = ru + s$. Also $rt \neq 0$ since both $r \neq 0$ and $t \neq 0$. Hence $\mathbf{AB} \in T$. Thus T is closed under matrix multiplication.

SG2 The identity element

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

of GL(2) is of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a = 1$ and $b = 0$. Thus $\mathbf{I} \in T$.

SG3 Let $\mathbf{A} \in T$. Then

$$\mathbf{A} = \begin{pmatrix} r & s \\ 0 & 1 \end{pmatrix},$$

for some $r, s \in \mathbb{R}$ with $r \neq 0$. The inverse of $\mathbf{A}$ in GL(2) is

$$\mathbf{A}^{-1} = \frac{1}{r} \begin{pmatrix} 1 & -s \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1/r & -s/r \\ 0 & 1 \end{pmatrix}.$$

This matrix is of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a = 1/r$ and $b = -s/r$. Also $1/r \neq 0$. So $\mathbf{A}^{-1} \in T$. Thus T contains the inverse of each of its elements.

Since the three subgroup properties hold, T is a subgroup of GL(2).

(d) The set V is a subset of the group GL(2), because each matrix

$$\begin{pmatrix} 1+n & n \\ -n & 1-n \end{pmatrix}$$

in V has determinant

$$(1+n)(1-n) - n(-n) = 1 - n^2 + n^2 = 1 \neq 0$$

and is therefore invertible. We show that the three subgroup properties hold for V.

SG1 Let $\mathbf{A}, \mathbf{B} \in V$. Then

$$\mathbf{A} = \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix}$$

and

$$\mathbf{B} = \begin{pmatrix} 1+b & b \\ -b & 1-b \end{pmatrix},$$

for some $a, b \in \mathbb{Z}$. So

$$\begin{aligned}
\mathbf{AB} &= \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \begin{pmatrix} 1+b & b \\ -b & 1-b \end{pmatrix} \\
&= \begin{pmatrix} (1+a)(1+b) + a(-b) & (1+a)b + a(1-b) \\ -a(1+b) + (1-a)(-b) & -ab + (1-a)(1-b) \end{pmatrix} \\
&= \begin{pmatrix} 1+a+b & a+b \\ -a-b & 1-a-b \end{pmatrix} \\
&= \begin{pmatrix} 1+(a+b) & a+b \\ -(a+b) & 1-(a+b) \end{pmatrix}.
\end{aligned}$$

This matrix is of the form

$$\begin{pmatrix} 1+n & n \\ -n & 1-n \end{pmatrix}$$

with $n = a+b$. Also $a+b \in \mathbb{Z}$ since $a \in \mathbb{Z}$ and $b \in \mathbb{Z}$. Hence $\mathbf{AB} \in V$. Thus V is closed under matrix multiplication.

SG2 The identity element

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

of GL(2) is of the form

$$\begin{pmatrix} 1+n & n \\ -n & 1-n \end{pmatrix}$$

with $n = 0$. So $\mathbf{I} \in V$.

SG3 Let $\mathbf{A} \in V$. Then

$$\mathbf{A} = \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix},$$

for some $a \in \mathbb{Z}$.

The inverse of $\mathbf{A}$ in GL(2) is

$$\begin{aligned}
\mathbf{A}^{-1} &= \frac{1}{(1+a)(1-a) + a^2} \begin{pmatrix} 1-a & -a \\ a & 1+a \end{pmatrix} \\
&= \begin{pmatrix} 1-a & -a \\ a & 1+a \end{pmatrix}.
\end{aligned}$$

This matrix is of the form

$$\begin{pmatrix} 1+n & n \\ -n & 1-n \end{pmatrix}$$

with $n = -a$. Also $-a \in \mathbb{Z}$ since $a \in \mathbb{Z}$. Hence $\mathbf{A}^{-1} \in V$. Thus V contains the inverse of each of its elements.

Since the three subgroup properties hold, T is a subgroup of GL(2).

Solution to Additional Exercise E2

Let the mapping ϕ be defined by

$$\begin{aligned}
\phi : [0,1) &\longrightarrow S^+(\bigcirc) \\
x &\longmapsto r_{2\pi x},
\end{aligned}$$

where for any angle θ the notation r_θ denotes the rotation through θ about the centre of the disc.

First we show that ϕ is one-to-one.

Let $x, y \in [0,1)$, and suppose that

$$\phi(x) = \phi(y),$$

that is,

$$r_{2\pi x} = r_{2\pi y}.$$

It follows that

$$2\pi x = 2\pi y + 2\pi n,$$

for some $n \in \mathbb{Z}$. This equation gives

$$x = y + n,$$

and since $x, y \in [0,1)$ and $n \in \mathbb{Z}$ we must have $n = 0$ and hence $x = y$. Thus ϕ is one-to-one.

Next we show that ϕ is onto. Consider any element of $S^+(\bigcirc)$. It can be written as r_θ for some $\theta \in [0, 2\pi)$. Let $x = \theta/(2\pi)$. Then $x \in [0,1)$ and $\theta = 2\pi x$ so

$$r_\theta = r_{2\pi x} = \phi(x).$$

Thus ϕ is onto.

Finally we show that

$$\phi(x +_1 y) = \phi(x) \circ \phi(y)$$

for all $x, y \in [0, 1)$.

Let $x, y \in [0, 1)$. Then

$$\begin{aligned}\phi(x +_1 y) &= r_{2\pi(x+_1 y)} \\ &= r_{2\pi(x+y-\lfloor x+y \rfloor)} \\ &= r_{2\pi x + 2\pi y - 2\pi\lfloor x+y \rfloor} \\ &= r_{2\pi x} \circ r_{2\pi y} \circ r_{-2\pi\lfloor x+y \rfloor} \\ &= r_{2\pi x} \circ r_{2\pi y} \quad (\text{since } \lfloor x+y \rfloor \in \mathbb{Z}) \\ &= \phi(x) \circ \phi(y),\end{aligned}$$

as required.

We have now shown that ϕ is an isomorphism. It follows that $\big([0,1), +_1\big) \cong (S^+(\bigcirc), \circ)$.

Solution to Additional Exercise E3

We use Strategy E1 for partitioning a group into left cosets of a subgroup. The left cosets are as follows.

$$\begin{aligned} H &= \{e,\ (1\ 3)\} \\ (1\ 2)H &= \{(1\ 2) \circ e,\ (1\ 2) \circ (1\ 3)\} \\ &= \{(1\ 2),\ (1\ 3\ 2)\} \\ (2\ 3)H &= \{(2\ 3) \circ e,\ (2\ 3) \circ (1\ 3)\} \\ &= \{(2\ 3),\ (1\ 2\ 3)\} \end{aligned}$$

In summary, the partition is

$$\{e,\ (1\ 3)\}, \quad \{(1\ 2),\ (1\ 3\ 2)\}, \quad \{(2\ 3),\ (1\ 2\ 3)\}.$$

Solution to Additional Exercise E4

(a) The group table of $(G, \circ)$ shows that w has order 2. Hence $H = \{e, w\}$ is the cyclic subgroup generated by w, so in particular it is a subgroup.

We use Strategy E1 to partition G into left cosets of H. The left cosets are as follows.

$$\begin{aligned} H &= \{e, w\} \\ iH &= \{i \circ e,\ i \circ w\} = \{i, x\} \\ jH &= \{j \circ e,\ j \circ w\} = \{j, y\} \\ kH &= \{k \circ e,\ k \circ w\} = \{k, z\} \end{aligned}$$

In summary, the partition is

$$\{e, w\}, \quad \{i, x\}, \quad \{j, y\}, \quad \{k, z\}.$$

(b) A Cayley table for H (obtained by crossing out the unwanted rows and columns of the group table for $(G, \circ)$) is as follows.

$\circ$	e	i	w	x
e	e	i	w	x
i	i	w	x	e
w	w	x	e	i
x	x	e	i	w

We check the three subgroup properties.

SG1 Every element in the body of the table is in H, so H is closed under function composition.

SG2 The identity element e of G is in H.

SG3 The Cayley table (or the original group table) shows that the elements e and w are self-inverse, and the elements i and x are inverses of each other. So H contains the inverse in G of each of its elements.

Hence H satisfies the three subgroup properties, and so is a subgroup of $(G, \circ)$.

Now we partition G into left cosets of H. One of the left cosets is the subgroup H:

$$\{e, i, w, x\}.$$

Since each left coset contains four elements, the other left coset must be

$$\{j, k, y, z\}.$$

Thus the partition is

$$\{e, i, w, x\}, \quad \{j, k, y, z\}.$$

Solution to Additional Exercise E5

(a) The left coset $(1\ 2)H$ consists of all the elements of S_4 of the form

$$(1\ 2) \circ h,$$

where $h \in H$. Let h be any element of H, so $h(1) = 1$, and consider the effect of the composite $(1\ 2) \circ h$ on the symbol 1:

$$1 \xrightarrow{h} 1 \xrightarrow{(1\ 2)} 2.$$

So each element of the left coset $(1\ 2)H$ maps 1 to 2.

(b) The left coset $(1\ 2)H$ contains six elements (because H has order 6) and there are exactly six elements of S_4 that map 1 to 2. So the left coset $(1\ 2)H$ is the set of such elements, which is

$$\{(1\,2),\ (1\,2)(3\,4),\ (1\,2\,3),\ (1\,2\,4),\ (1\,2\,3\,4),\ (1\,2\,4\,3)\}.$$

(c) Similarly, the left coset $(1\ 3)H$ is the set of elements of S_4 that map 1 to 3, which is

$$\{(1\,3), (1\,3)(2\,4), (1\,3\,2), (1\,3\,4), (1\,3\,2\,4), (1\,3\,4\,2)\},$$

and the left coset $(1\ 4)H$ is the set of elements of S_4 that map 1 to 4, which is

$$\{(1\,4), (1\,4)(2\,3), (1\,4\,2), (1\,4\,3), (1\,4\,2\,3), (1\,4\,3\,2)\}.$$

Solution to Additional Exercise E6

(a) We use Strategy E1. The first three cosets are as follows.

$$\begin{aligned} H &= \{0, 9\} \\ 1 + H &= \{1 +_{18} 0,\ 1 +_{18} 9\} = \{1, 10\} \\ 2 + H &= \{2 +_{18} 0,\ 2 +_{18} 9\} = \{2, 11\} \end{aligned}$$

Continuing in the same way, using the elements 3, 4, 5, 6, 7 and 8 in turn, we obtain the following further cosets:

$$\begin{gathered} \{3, 12\}, \quad \{4, 13\}, \quad \{5, 14\}, \\ \{6, 15\}, \quad \{7, 16\}, \quad \{8, 17\}. \end{gathered}$$

In summary, the partition is

$$\begin{gathered} \{0, 9\}, \quad \{1, 10\}, \quad \{2, 11\}, \quad \{3, 12\}, \quad \{4, 13\}, \\ \{5, 14\}, \quad \{6, 15\}, \quad \{7, 16\}, \quad \{8, 17\}. \end{gathered}$$

(b) We use Strategy E1. The first three cosets are as follows.

$$\begin{aligned} H &= \{0, 6, 12\} \\ 1 + H &= \{1 +_{18} 0,\ 1 +_{18} 6,\ 1 +_{18} 12\} = \{1, 7, 13\} \\ 2 + H &= \{2 +_{18} 0,\ 2 +_{18} 6,\ 2 +_{18} 12\} = \{2, 8, 14\} \end{aligned}$$

Continuing this process, using the elements 3, 4 and 5 in turn, we obtain the following further cosets:

$$\{3, 9, 15\}, \quad \{4, 10, 16\}, \quad \{5, 11, 17\}.$$

In summary, the partition is

$$\begin{gathered} \{0, 6, 12\}, \quad \{1, 7, 13\}, \quad \{2, 8, 14\}, \\ \{3, 9, 15\}, \quad \{4, 10, 16\}, \quad \{5, 11, 17\}. \end{gathered}$$

Solution to Additional Exercise E7

We use Strategy E1. The first three cosets are as follows.

$$\begin{aligned} H &= \{1, 12\} \\ 2H &= 2\{1, 12\} = \{2 \times_{13} 1,\ 2 \times_{13} 12\} = \{2, 11\} \\ 3H &= 3\{1, 12\} = \{3 \times_{13} 1,\ 3 \times_{13} 12\} = \{3, 10\} \end{aligned}$$

Continuing this process, using the elements 4, 5 and 6 in turn, we obtain the following further cosets:

$$4H = \{4, 9\}, \quad 5H = \{5, 8\}, \quad 6H = \{6, 7\}.$$

In summary, the partition is

$$\{1, 12\}, \quad \{2, 11\}, \quad \{3, 10\}, \quad \{4, 9\}, \quad \{5, 8\}, \quad \{6, 7\}.$$

Solution to Additional Exercise E8

The partition into cosets is

$$\begin{aligned} 6\mathbb{Z} &= \{\ldots, -12, -6, 0, 6, 12, \ldots\}, \\ 1 + 6\mathbb{Z} &= \{\ldots, -11, -5, 1, 7, 13, \ldots\}, \\ 2 + 6\mathbb{Z} &= \{\ldots, -10, -4, 2, 8, 14, \ldots\}, \\ 3 + 6\mathbb{Z} &= \{\ldots, -9, -3, 3, 9, 15, \ldots\}, \\ 4 + 6\mathbb{Z} &= \{\ldots, -8, -2, 4, 10, 16, \ldots\}, \\ 5 + 6\mathbb{Z} &= \{\ldots, -7, -1, 5, 11, 17, \ldots\}. \end{aligned}$$

Solution to Additional Exercise E9

(a) By the solution to Additional Exercise E4, the partition of G into *left* cosets of $H = \{e, w\}$ is

$$\{e, w\}, \quad \{i, x\}, \quad \{j, y\}, \quad \{k, z\}.$$

To obtain the partition into right cosets of H, we replace each element by its inverse.

The group table of $(G, \circ)$ shows that the elements e and w are self-inverse and the other elements form the following pairs of inverses:

$$i, x; \quad j, y; \quad k, z.$$

Hence the partition into right cosets is

$$\{e, w\}, \quad \{x, i\}, \quad \{y, j\}, \quad \{z, k\},$$

that is,

$$\{e, w\}, \quad \{i, x\}, \quad \{j, y\}, \quad \{k, z\}.$$

(This is the same as the partition into left cosets of $H = \{e, w\}$.)

(b) One of the right cosets is the subgroup H:

$$\{e, i, w, x\}.$$

Since each right coset contains four elements, the other right coset must be

$$\{j, k, y, z\}.$$

Thus the partition is

$$\{e, i, w, x\}, \quad \{j, k, y, z\}.$$

(Again this is the same as the partition into left cosets of $H = \{e, i, w, x\}$.)

Solution to Additional Exercise E10

By the solution to Additional Exercise E5, the partition of S_4 into *left* cosets of H is

$\{e, (2\,3), (2\,4), (3\,4), (2\,3\,4), (2\,4\,3)\}$,

$\{(1\,2), (1\,2)(3\,4), (1\,2\,3), (1\,2\,4), (1\,2\,3\,4), (1\,2\,4\,3)\}$,

$\{(1\,3), (1\,3)(2\,4), (1\,3\,2), (1\,3\,4), (1\,3\,2\,4), (1\,3\,4\,2)\}$,

$\{(1\,4), (1\,4)(2\,3), (1\,4\,2), (1\,4\,3), (1\,4\,2\,3), (1\,4\,3\,2)\}$.

To obtain the partition of S_4 into right cosets of H, we replace each element by its inverse. Hence the partition into right cosets is

$\{e, (2\,3), (2\,4), (3\,4), (2\,4\,3), (2\,3\,4)\}$,

$\{(1\,2), (1\,2)(3\,4), (1\,3\,2), (1\,4\,2), (1\,4\,3\,2), (1\,3\,4\,2)\}$,

$\{(1\,3), (1\,3)(2\,4), (1\,2\,3), (1\,4\,3), (1\,4\,2\,3), (1\,2\,4\,3)\}$,

$\{(1\,4), (1\,4)(2\,3), (1\,2\,4), (1\,3\,4), (1\,3\,2\,4), (1\,2\,3\,4)\}$.

Solution to Additional Exercise E11

(a) By the solutions to Additional Exercises E4(a) and E9(a), the partition of $(G, \circ)$ into left cosets of the subgroup $H = \{e, w\}$ is the same as the partition of $(G, \circ)$ into right cosets of this subgroup. Therefore this is a normal subgroup.

(b) By the solutions to Additional Exercises E4(b) and E9(b), the partition of $(G, \circ)$ into left cosets of the subgroup $H = \{e, i, w, x\}$ is the same as the partition of $(G, \circ)$ into right cosets of this subgroup. Therefore this is a normal subgroup.

(Alternatively, this subgroup is a normal subgroup because it has index 2.)

Solution to Additional Exercise E12

The solutions to Additional Exercises E5 and E10 show that the partition of S_4 into left cosets of H is not the same as the partition of S_4 into right cosets of H. For example, the permutations $(1\ 2)$ and $(1\ 2\ 3)$ lie in the same left coset of H, but in different right cosets of H. Therefore H is not a normal subgroup of S_4.

Solution to Additional Exercise E13

(a) The group $\mathbb{Z}_{16}$ is abelian, so all its subgroups are normal. In particular, $H = \langle 4 \rangle = \{0, 4, 8, 12\}$ is a normal subgroup of $\mathbb{Z}_{16}$.

(b) The group A_5 is a subgroup of index 2 in S_5, and is therefore a normal subgroup of S_5. (Alternatively, this follows from Corollary E12 in Unit E1.)

Solution to Additional Exercise E14

First we have to show that H is a *subgroup* of $(G, \circ)$. A Cayley table for H (obtained by crossing out the unwanted rows and columns of the group table for $(G, \circ)$) is as follows.

$\circ$	e	k	w	z
e	e	k	w	z
k	k	w	z	e
w	w	z	e	k
z	z	e	k	w

We check the three subgroup properties.

SG1 Every element in the body of the table is in H, so H is closed under $\circ$.

SG2 The identity element e of G is in H.

SG3 The Cayley table shows that the elements e and w are self-inverse, and the elements k and z are inverses of each other. So H contains the inverse of each of its elements.

Hence H satisfies the three subgroup properties, and so is a subgroup of $(G, \circ)$.

Now we have to show that H is normal in $(G, \circ)$. It is, because it has index 2 in $(G, \circ)$.

Thus H is a normal subgroup of $(G, \circ)$.

Solution to Additional Exercise E15

(a) The set $\{e, b\}$ is a subgroup of G because b has order 2, so this set is the subgroup generated by b.

To show that $\{e, b\}$ is normal in G, we determine the partition of G into left cosets of N and the partition of G into right cosets of N.

The left cosets of $\{e, b\}$ in G are

$$\begin{aligned} &\{e, b\}, \\ a\{e, b\} &= \{a, c\}, \\ w\{e, b\} &= \{w, y\}, \\ x\{e, b\} &= \{x, z\}. \end{aligned}$$

So the partition of G into left cosets of $\{e, b\}$ is

$$\{e, b\}, \quad \{a, c\}, \quad \{w, y\}, \quad \{x, z\}.$$

The group table of $(G, \circ)$ shows that the elements e and b are self-inverse and the other elements form the following pairs of inverses:

$$a, c; \quad w, y; \quad x, z.$$

Replacing the elements in the left cosets of $\{e, b\}$ by their inverses gives the partition of G into right cosets of $\{e, b\}$, as follows:

$$\{e, b\}, \quad \{c, a\}, \quad \{y, w\}, \quad \{z, x\}.$$

The two partitions are the same, so $\{e, b\}$ is a normal subgroup of G.

(b) To show that $\{e, w, b, y\}$ is a normal subgroup of $(G, \circ)$, first we have to show that it is a subgroup of $(G, \circ)$.

A Cayley table for $\{e, w, b, y\}$ (obtained by crossing out the unwanted rows and columns of the group table for $(G, \circ)$) is as follows.

	e	w	b	y
e	e	w	b	y
w	w	b	y	e
b	b	y	e	w
y	y	e	w	b

We check the three subgroup properties.

SG1 Every element in the body of the table is in $\{e, w, b, y\}$, so $\{e, w, b, y\}$ is closed under $\circ$.

SG2 The identity element e of G is in $\{e, w, b, y\}$.

SG3 We know that the elements e and b are self-inverse, and the elements w and y are inverses of each other. So $\{e, w, b, y\}$ contains the inverse of each of its elements.

Hence $\{e, w, b, y\}$ satisfies the three subgroup properties, and so is a subgroup of $(G, \circ)$.

Also, the subgroup $\{e, w, b, y\}$ has index 2 in G, so it is a normal subgroup of G.

Additional exercises for Unit E2

Section 1

Additional Exercise E16

The following table is the group table of a group $(G, \circ)$.

$\circ$	e	i	j	k	w	x	y	z
e	e	i	j	k	w	x	y	z
i	i	w	k	y	x	e	z	j
j	j	z	w	i	y	k	e	x
k	k	j	x	w	z	y	i	e
w	w	x	y	z	e	i	j	k
x	x	e	z	j	i	w	k	y
y	y	k	e	x	j	z	w	i
z	z	y	i	e	k	j	x	w

The solutions to Additional Exercises E9 and E11 show that $H = \{e, w\}$ is a normal subgroup of G, with cosets

$$\begin{aligned} H &= \{e, w\}, \\ iH &= \{i, x\}, \\ jH &= \{j, y\}, \\ kH &= \{k, z\}. \end{aligned}$$

(a) By using the definition of set composition, determine the following set composites and verify that each of them is equal to one of the cosets of H in G.

  (i) $iH \cdot jH$

  (ii) $jH \cdot jH$

  (iii) $iH \cdot (jH \cdot kH)$

(b) Use the rule for composing cosets from Theorem E14 of Unit E2 to check your answers to part (a).

Additional Exercise E17

Let G and H be the same group and normal subgroup as in Additional Exercise E16 above. The cosets of H in G are given in the same additional exercise.

(a) Construct the group table of the quotient group G/H.

(b) State a standard group that is isomorphic to this quotient group.

Additional Exercise E18

Consider the group $(\mathbb{Z}_{13}^*, \times_{13})$ and its subgroup $H = \langle 12 \rangle = \{1, 12\}$. By the solution to Additional Exercise E7, the cosets of H in $\mathbb{Z}_{13}^*$ are

$$\{1, 12\}, \quad \{2, 11\}, \quad \{3, 10\}, \quad \{4, 9\}, \quad \{5, 8\}, \quad \{6, 7\}.$$

(a) Explain why H is a normal subgroup of $\mathbb{Z}_{13}^*$.

(b) Construct the group table of the quotient group $\mathbb{Z}_{13}^*/H$.

(c) State a standard group that is isomorphic to this quotient group.

Additional Exercise E19

Consider the subset $N = \{1, 7\}$ of the group U_{16}.

(a) List the elements of U_{16}.

(b) Show that $N = \{1, 7\}$ is a normal subgroup of the group U_{16}.

(c) Find the cosets of N in U_{16}.

(d) Construct a group table for the quotient group U_{16}/N.

(e) State the inverse of each of the elements of the quotient group.

(f) State a standard group that is isomorphic to this quotient group.

Additional Exercise E20

Consider the subset $N = \{0, 3\}$ of the group $\mathbb{Z}_6$.

(a) Show that N is a normal subgroup of $\mathbb{Z}_6$.

(b) Find the cosets of N in $\mathbb{Z}_6$.

(c) Construct a group table for the quotient group $\mathbb{Z}_6/N$.

(d) State a standard group that is isomorphic to this quotient group.

Additional Exercise E21

(a) Construct the group table for the quotient group $\mathbb{Z}/3\mathbb{Z}$.

(b) State a standard group that is isomorphic to this quotient group.

Section 2

Additional Exercise E22

Find all the elements of S_5 that conjugate $(1\ 3\ 5\ 2\ 4)$ to $(1\ 4\ 2\ 5\ 3)$.

Additional Exercise E23

Write down an element h of S_5 that conjugates $(1\ 2\ 3)(4\ 5)$ to $(1\ 3\ 5)(2\ 4)$ and an element g of S_5 that conjugates $(1\ 3\ 5)(2\ 4)$ to $(1\ 2)(3\ 4\ 5)$. Find $g \circ h$ and verify that it conjugates $(1\ 2\ 3)(4\ 5)$ to $(1\ 2)(3\ 4\ 5)$.

Additional Exercise E24

The following table is the group table of a group G. Determine the partition of G into conjugacy classes.

	e	a	b	c	w	x	y	z
e	e	a	b	c	w	x	y	z
a	a	b	c	e	x	y	z	w
b	b	c	e	a	y	z	w	x
c	c	e	a	b	z	w	x	y
w	w	z	y	x	b	a	e	c
x	x	w	z	y	c	b	a	e
y	y	x	w	z	e	c	b	a
z	z	y	x	w	a	e	c	b

Additional Exercise E25

Let x be an element of a group G. Prove that the subset

$$C(x) = \{g \in G : gxg^{-1} = x\}$$

of G is a subgroup of G.

($C(x)$ is the set of all elements of G that conjugate x to itself.)

Additional Exercise E26

Determine the conjugacy class containing π in each of the following groups.

(a) $(\mathbb{R}, +)$ (b) $(\mathbb{R}^*, \times)$

Section 3

Additional Exercise E27

Find all the subgroups of $S(\square)$ that are conjugate to the subgroup $\langle r\rangle = \{e, r\}$.

Additional Exercise E28

(a) Let G be a group and let H be the cyclic subgroup of G generated by the element $h \in G$; that is, $H = \langle h\rangle$.

Show that gHg^{-1} is the cyclic subgroup of G generated by the element ghg^{-1}; that is, $gHg^{-1} = \langle ghg^{-1}\rangle$.

Hint: To show that $gHg^{-1} = \langle ghg^{-1}\rangle$, show that $gHg^{-1} \subseteq \langle ghg^{-1}\rangle$ and $\langle ghg^{-1}\rangle \subseteq gHg^{-1}$.

(b) Hence find all the subgroups of S_4 that are conjugate to the subgroup

$$\{e, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\}.$$

Additional Exercise E29

In Worked Exercise B18 in Subsection 1.2 of Unit B2 you met the group $(X, *)$ where X is the subset of $\mathbb{R}^2$ consisting of all the points not on the y-axis, that is,

$$X = \{(a, b) \in \mathbb{R}^2 : a \neq 0\},$$

and $*$ is the binary operation on X defined by

$$(a, b) * (c, d) = (ac, ad + b).$$

It was shown that in this group the identity element is $(1, 0)$ and the inverse of the element (a, b) is $\left(\frac{1}{a}, -\frac{b}{a}\right)$.

In Additional Exercise B22(a) you saw that the set

$$C = \{(a, 0) : a \in \mathbb{R}^*\}$$

is a subgroup of this group X.

(a) Show that C is not a normal subgroup of X.

(b) Determine the following conjugate subgroups of C.

(i) $(1, 2)C(1, 2)^{-1}$ (ii) $(2, 1)C(2, 1)^{-1}$

Additional Exercise E30 Challenging

Let H and N be subgroups of a group G, with N normal in G. By Theorem B81 in Unit B4, $H \cap N$ is a subgroup of G.

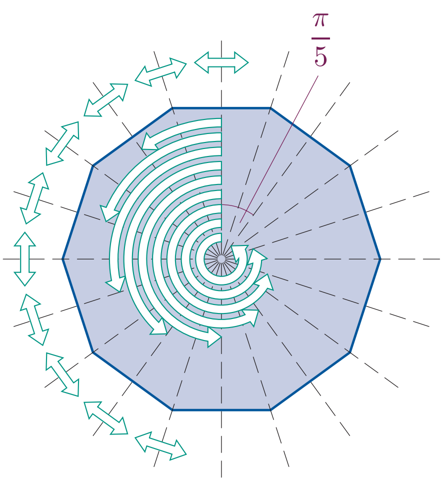
(a) Show that $H \cap N$ is a normal subgroup of H.

(b) Give an example to show that $H \cap N$ is not necessarily a normal subgroup of G.

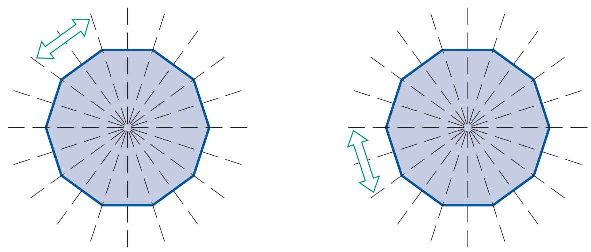
Section 4

Additional Exercise E31

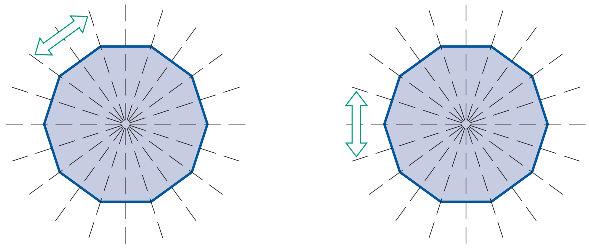
This exercise is about conjugacy in S(decagon), the symmetry group of the regular decagon, whose non-identity elements are shown below. In each of parts (a)–(d), use the ideas in the box headed 'Conjugate elements in the symmetry group of a figure' in Subsection 4.1 of Unit E2 to decide whether the two symmetries shown are conjugate in S(decagon). In each case, if the two symmetries are conjugate in S(decagon) then describe a symmetry in S(decagon) that conjugates the first symmetry in the pair to the second.



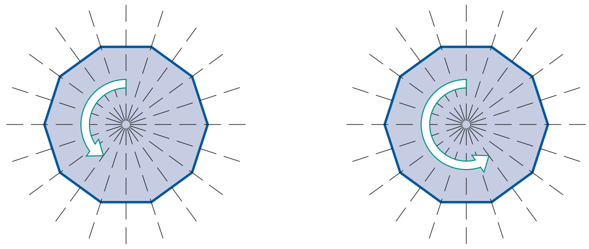
(a) The reflections in the axes shown below.



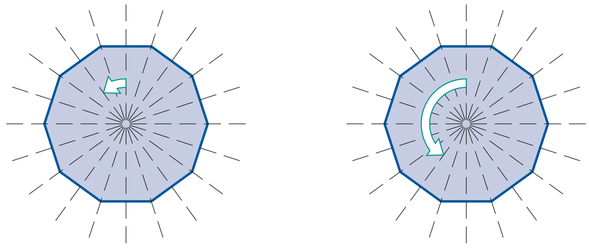
(b) The reflections in the axes shown below.



(c) The rotations shown below.



(d) The rotations shown below.

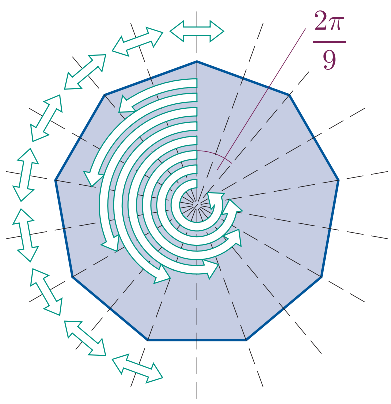


Additional Exercise E32

Do each of parts (a) and (b) below using only the following fact from Subsection 4.1 of Unit E2: two symmetries x and y of a figure F are conjugate in $S(F)$ if and only if there is a symmetry g of F that transforms a diagram illustrating x into a diagram illustrating y.

(a) Describe the conjugacy classes of the symmetry group of the regular decagon, describing the symmetries in each class geometrically. (See the diagram illustrating S(decagon) in Additional Exercise E31.)

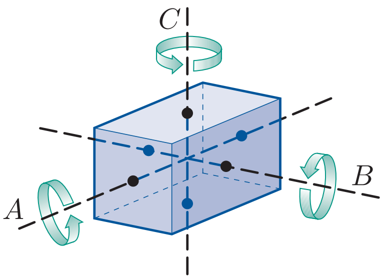
(b) Do the same for the symmetry group of the regular nonagon. (See the diagram below.)

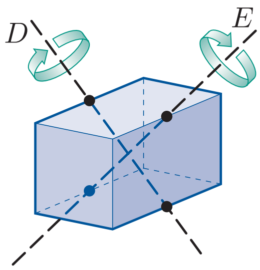


Additional Exercise E33

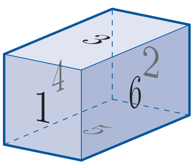
In this exercise you are asked to consider conjugacy in the group of *direct* symmetries of a solid figure, not in the whole symmetry group of the figure as in the worked exercises and exercises in Section 4 of Unit E2.

The group $S^+(P)$ of direct symmetries of the square prism P shown below has order 8. The prism has five axes of symmetry, as shown.





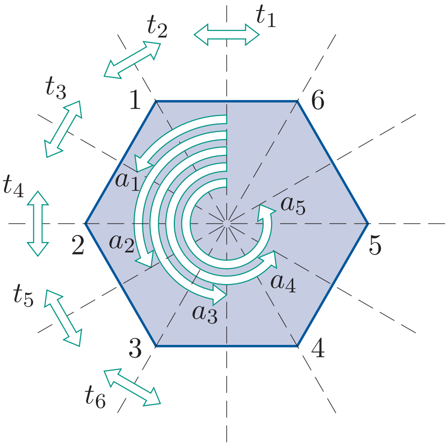
(a) Express the eight symmetries in $S^+(P)$ as permutations of the face labels $1, 2, 3, 4, 5, 6$ shown below, and describe these symmetries geometrically, referring to the axes of rotation by using the labels above.



(b) Since $S^+(P)$ is a subgroup of S_6, any symmetries that are conjugate in $S^+(P)$ are also conjugate in S_6 and hence have the same cycle structure. By starting with the partition of $S^+(P)$ by cycle structure, or otherwise, find the conjugacy classes of $S^+(P)$.

Additional Exercise E34

Consider $S(\hexagon)$, the symmetry group of the regular hexagon, where the non-identity symmetries are denoted as shown in the diagram below (and the identity symmetry is denoted by e, as usual).



(a) By using the solution to Exercise E93(a) in Subsection 4.2 of Unit E2, or otherwise, write down the conjugacy classes of $S(\hexagon)$ using the notation above for the symmetries.

(b) Use Strategy E5 from Unit E2 to determine all the normal subgroups of $S(\hexagon)$.

Section 5

Additional Exercise E35

Recall that the group D of all invertible 2×2 diagonal matrices and the group L of all invertible 2×2 lower triangular matrices are given by

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in \mathbb{R},\ ad \neq 0 \right\},$$

$$L = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : a, c, d \in \mathbb{R},\ ad \neq 0 \right\}.$$

Find the conjugate subgroup

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} D \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Is it equal to L? Justify your answer.

Additional Exercise E36

The following set is a subgroup of GL(2) (you may assume this):

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in \mathbb{R}^* \right\}.$$

Find the conjugate subgroup $\mathbf{B}H\mathbf{B}^{-1}$ for each of the following matrices $\mathbf{B}$.

(a) $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$

Additional Exercise E37

The solution to Additional Exercise E1(d) shows that the set

$$V = \left\{ \begin{pmatrix} 1+n & n \\ -n & 1-n \end{pmatrix} : n \in \mathbb{Z} \right\}$$

is a subgroup of GL(2).

Find the conjugate subgroup $\mathbf{B}V\mathbf{B}^{-1}$ for each of the following matrices $\mathbf{B}$. In each case determine whether $\mathbf{B}V\mathbf{B}^{-1}$ is equal to V.

(a) $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ (b) $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

Additional Exercise E38

The solution to Additional Exercise E1(a) shows that the set

$$R = \left\{ \begin{pmatrix} a & 0 \\ c & a \end{pmatrix} : a, c \in \mathbb{R},\ a \neq 0 \right\}$$

is a subgroup of GL(2).

Show that this subgroup R is a normal subgroup of the group L of invertible 2×2 lower triangular matrices under matrix multiplication.

Additional Exercise E39 Challenging

In Worked Exercise E36(a) in Subsection 5.2 of Unit E2 you saw that the set

$$M = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}$$

is a normal subgroup of the group U of all invertible 2×2 upper triangular matrices.

(a) Show that every coset of M in U can be expressed as

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M$$

for some $x \in \mathbb{R}^*$.

(b) Show that for each coset of M in U, there is a *unique* value of $x \in \mathbb{R}^*$ such that the coset can be expressed as

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M.$$

(c) Show that the quotient group U/M is isomorphic to the group $(\mathbb{R}^*, \times)$.

Hint: You may find it easier to work with an isomorphism from $(\mathbb{R}^*, \times)$ to U/M rather than from U/M to $(\mathbb{R}^*, \times)$.

Additional Exercise E40 Challenging

The solution to Exercise E98 in Subsection 5.2 of Unit E2 shows that the set

$$S = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\}$$

is a normal subgroup of the group U of all invertible 2×2 upper triangular matrices.

(a) Show that every coset of S in U can be expressed as

$$\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} S$$

for some $p, s \in \mathbb{R}^*$.

(b) Show that for each coset of S in U, there is a *unique* pair of numbers $p, s \in \mathbb{R}^*$ such that the coset can be expressed as

$$\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} S.$$

(c) Complete the following formula for the binary operation on U/S:

$$\begin{pmatrix} p_1 & 0 \\ 0 & s_1 \end{pmatrix} S \cdot \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix} S = \begin{pmatrix} ? & 0 \\ 0 & ? \end{pmatrix} S.$$

Solutions to additional exercises for Unit E2

Solution to Additional Exercise E16

(a) **(i)** $$\begin{aligned} iH \cdot jH &= \{i, x\} \cdot \{j, y\} \\ &= \{i \circ j,\ i \circ y,\ x \circ j,\ x \circ y\} \\ &= \{k, z, z, k\} \\ &= \{k, z\} \\ &= kH \end{aligned}$$

(ii) $$\begin{aligned} jH \cdot jH &= \{j, y\} \cdot \{j, y\} \\ &= \{j \circ j,\ j \circ y,\ y \circ j,\ y \circ y\} \\ &= \{w, e, e, w\} \\ &= \{e, w\} \\ &= H \end{aligned}$$

(iii) $$\begin{aligned} jH \cdot kH &= \{j, y\} \cdot \{k, z\} \\ &= \{j \circ k,\ j \circ z,\ y \circ k,\ y \circ z\} \\ &= \{i, x, x, i\} \\ &= \{i, x\}, \end{aligned}$$

so

$$\begin{aligned} iH \cdot (jH \cdot kH) &= \{i, x\} \cdot \{i, x\} \\ &= \{i \circ i,\ i \circ x,\ x \circ i,\ x \circ x\} \\ &= \{w, e, e, w\} \\ &= \{e, w\} \\ &= H. \end{aligned}$$

(b) The rule for composing cosets of H in G is

$$xH \cdot yH = (x \circ y)H \quad \text{for all } x, y \in G.$$

This gives the following.

(i) $iH \cdot jH = (i \circ j)H = kH$

(ii) $jH \cdot jH = (j \circ j)H = wH = H$

(iii) $$\begin{aligned} iH \cdot (jH \cdot kH) &= iH \cdot (j \circ k)H \\ &= iH \cdot iH \\ &= (i \circ i)H \\ &= wH \\ &= H \end{aligned}$$

These answers agree with those found in part (a).

Solution to Additional Exercise E17

(a) The rule for composing cosets of H in G is

$$xH \cdot yH = (x \circ y)H \quad \text{for all } x, y \in G.$$

We express each composite coset consistently, as one of H, iH, jH or kH.

For example,

$$\begin{aligned} iH \cdot jH &= (i \circ j)H = kH, \\ jH \cdot iH &= (j \circ i)H = zH = kH \quad (\text{since } z \in kH), \\ kH \cdot kH &= (k \circ k)H = wH = H \quad (\text{since } w \in H). \end{aligned}$$

We thus obtain the following group table.

$\cdot$	H	iH	jH	kH
H	H	iH	jH	kH
iH	iH	H	kH	jH
jH	jH	kH	H	iH
kH	kH	jH	iH	H

(b) Since G/H has order 4 and each of its elements is self-inverse, it is isomorphic to the Klein four-group V (and to $S(\square)$).

Solution to Additional Exercise E18

(a) The group $\mathbb{Z}_{13}^*$ is abelian, so all of its subgroups are normal; in particular, H is normal.

(b) The rule for composing cosets of H in $\mathbb{Z}_{13}^*$ is

$$aH \cdot bH = (a \times_{13} b)H.$$

For example,

$$4H \cdot 5H = (4 \times_{13} 5)H = 7H = 6H \quad (\text{since } 7 \in 6H).$$

We thus obtain the following group table.

$\cdot$	H	$2H$	$3H$	$4H$	$5H$	$6H$
H	H	$2H$	$3H$	$4H$	$5H$	$6H$
$2H$	$2H$	$4H$	$6H$	$5H$	$3H$	H
$3H$	$3H$	$6H$	$4H$	H	$2H$	$5H$
$4H$	$4H$	$5H$	H	$3H$	$6H$	$2H$
$5H$	$5H$	$3H$	$2H$	$6H$	H	$4H$
$6H$	$6H$	H	$5H$	$2H$	$4H$	$3H$

(c) The quotient group $\mathbb{Z}_{13}^*/H$ is an abelian group of order 6, so it is isomorphic to C_6 (and to $\mathbb{Z}_6$).

Solution to Additional Exercise E19

(a) We have

$$U_{16} = \{1, 3, 5, 7, 9, 11, 13, 15\}.$$

(b) In U_{16} we have

$$7^2 = 7 \times_{16} 7 = 1,$$

so

$$\langle 7 \rangle = \{1, 7\} = N.$$

Hence N is a subgroup. Also, N is normal in U_{16} because U_{16} is abelian.

(c) The cosets of N in U_{16} are

$$\begin{aligned} N &= \{1, 7\}, \\ 3N &= \{3 \times_{16} 1,\ 3 \times_{16} 7\} = \{3, 5\}, \\ 9N &= \{9 \times_{16} 1,\ 9 \times_{16} 7\} = \{9, 15\}, \\ 11N &= \{11 \times_{16} 1,\ 11 \times_{16} 7\} = \{11, 13\}. \end{aligned}$$

(Remember to use congruence properties to calculate products like those above and those required for part (d) quickly and accurately. For example,

$$\begin{aligned} 11 \times 7 &\equiv (-5) \times 7 \\ &\equiv -35 \\ &\equiv -3 \\ &\equiv 13 \pmod{16} \end{aligned}$$

and

$$\begin{aligned} 9 \times 9 &\equiv 3 \times 27 \\ &\equiv 3 \times 11 \\ &\equiv 33 \\ &\equiv 1 \pmod{16}.) \end{aligned}$$

(d) A group table for the quotient group is as follows.

$\cdot$	N	$3N$	$9N$	$11N$
N	N	$3N$	$9N$	$11N$
$3N$	$3N$	$9N$	$11N$	N
$9N$	$9N$	$11N$	N	$3N$
$11N$	$11N$	N	$3N$	$9N$

(e) The elements N (the identity) and $9N$ are self-inverse, and the elements $3N$ and $11N$ are inverses of each other.

(f) The quotient group U_{16}/N has four elements, exactly two of which are self-inverse, so it is isomorphic to the cyclic group C_4 (and to $\mathbb{Z}_4$).

Solution to Additional Exercise E20

(a) In the (additive) group $\mathbb{Z}_6$ we have

$$3 +_6 3 = 0,$$

so

$$\langle 3 \rangle = \{0, 3\} = N.$$

Hence N is a subgroup. Also, N is normal in $\mathbb{Z}_6$ because $\mathbb{Z}_6$ is abelian.

(b) The cosets of N in $\mathbb{Z}_6$ are

$$\begin{aligned} N &= \{0, 3\}, \\ 1 + N &= \{1 +_6 0,\ 1 +_6 3\} = \{1, 4\}, \\ 2 + N &= \{2 +_6 1,\ 2 +_6 3\} = \{2, 5\}. \end{aligned}$$

(c) A group table for the quotient group is as follows.

$+$	N	$1+N$	$2+N$
N	N	$1+N$	$2+N$
$1+N$	$1+N$	$2+N$	N
$2+N$	$2+N$	N	$1+N$

(d) The quotient group $\mathbb{Z}_6/N$ has three elements, so it is isomorphic to the cyclic group C_3 (and to $\mathbb{Z}_3$).

Solution to Additional Exercise E21

(a) There are three cosets of $3\mathbb{Z}$ in $\mathbb{Z}$, namely

$$\begin{aligned} 3\mathbb{Z} &= \{\ldots, -6, -3, 0, 3, 6, \ldots\}, \\ 1 + 3\mathbb{Z} &= \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \\ 2 + 3\mathbb{Z} &= \{\ldots, -4, -1, 2, 5, 8, \ldots\}. \end{aligned}$$

These three cosets are the elements of the quotient group $\mathbb{Z}/3\mathbb{Z}$. The binary operation of $\mathbb{Z}/3\mathbb{Z}$ is given by

$$(x + 3\mathbb{Z}) + (y + 3\mathbb{Z}) = (x +_3 y) + 3\mathbb{Z}.$$

The group table of $\mathbb{Z}/3\mathbb{Z}$ is therefore as follows.

$+$	$3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$3\mathbb{Z}$	$3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$
$1+3\mathbb{Z}$	$1+3\mathbb{Z}$	$2+3\mathbb{Z}$	$3\mathbb{Z}$
$2+3\mathbb{Z}$	$2+3\mathbb{Z}$	$3\mathbb{Z}$	$1+3\mathbb{Z}$

(b) The quotient group $\mathbb{Z}/3\mathbb{Z}$ has three elements, so it is isomorphic to the cyclic group C_3 (and to $\mathbb{Z}_3$).

(See also Theorem E16 in Unit E2.)

Solution to Additional Exercise E22

Using the five possible starting symbols for the 5-cycle $(1\ 4\ 2\ 5\ 3)$, we obtain

$$\begin{array}{ll} & (1\ 3\ 5\ 2\ 4) \\ g & \downarrow\downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g=(2\ 5)(3\ 4), \\ & (1\ 4\ 2\ 5\ 3) \end{array}$$

$$\begin{array}{ll} & (1\ 3\ 5\ 2\ 4) \\ g & \downarrow\downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g=(1\ 4)(2\ 3), \\ & (4\ 2\ 5\ 3\ 1) \end{array}$$

$$\begin{array}{ll} & (1\ 3\ 5\ 2\ 4) \\ g & \downarrow\downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g=(1\ 2)(3\ 5), \\ & (2\ 5\ 3\ 1\ 4) \end{array}$$

$$\begin{array}{ll} & (1\ 3\ 5\ 2\ 4) \\ g & \downarrow\downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g=(1\ 5)(2\ 4), \\ & (5\ 3\ 1\ 4\ 2) \end{array}$$

$$\begin{array}{ll} & (1\ 3\ 5\ 2\ 4) \\ g & \downarrow\downarrow\downarrow\downarrow\downarrow\ \text{, which gives } g=(1\ 3)(4\ 5). \\ & (3\ 1\ 4\ 2\ 5) \end{array}$$

Solution to Additional Exercise E23

Writing

$$\begin{array}{ll} & (1\ 2\ 3)(4\ 5) \\ h & \downarrow\downarrow\downarrow\ \ \downarrow\downarrow \\ & (1\ 3\ 5)(2\ 4) \end{array}$$

and

$$\begin{array}{ll} & (1\ 3\ 5)(2\ 4) \\ g & \downarrow\downarrow\downarrow\ \ \downarrow\downarrow \\ & (3\ 4\ 5)(1\ 2) \end{array}$$

gives the conjugating elements

$$h=(2\ 3\ 5\ 4) \quad \text{and} \quad g=(1\ 3\ 4\ 2).$$

We have

$$g\circ h=(1\ 3\ 4\ 2)\circ(2\ 3\ 5\ 4)=(1\ 3\ 5\ 2\ 4),$$

so conjugating $(1\ 2\ 3)(4\ 5)$ by $g\circ h$ gives

$$\begin{array}{ll} & (1\ 2\ 3)(4\ 5) \\ g\circ h & \downarrow\downarrow\downarrow\ \ \downarrow\downarrow \\ & (3\ 4\ 5)(1\ 2)=(1\ 2)(3\ 4\ 5). \end{array}$$

That is, $g\circ h$ conjugates $(1\ 2\ 3)(4\ 5)$ to $(1\ 2)(3\ 4\ 5)$, as expected.

(You can see why this is by putting the renaming diagrams for g and h above together:

$$\begin{array}{ll} & (1\ 2\ 3)(4\ 5) \\ h & \downarrow\downarrow\downarrow\ \ \downarrow\downarrow \\ & (1\ 3\ 5)(2\ 4) \\ g & \downarrow\downarrow\downarrow\ \ \downarrow\downarrow \\ & (3\ 4\ 5)(1\ 2). \end{array}$$

This diagram shows that $g\circ h$ renames $(1\ 2\ 3)(4\ 5)$ to $(1\ 2)(3\ 4\ 5)$.)

(There are many other possible answers for h and g, as follows.

The remaining ways of writing $(1\ 3\ 5)(2\ 4)$ give the following conjugating elements h.

$(1\ 3\ 5)(4\ 2)$ gives $h=(2\ 3\ 5)$,
$(3\ 5\ 1)(2\ 4)$ gives $h=(1\ 3)(2\ 5\ 4)$,
$(3\ 5\ 1)(4\ 2)$ gives $h=(1\ 3)(2\ 5)$,
$(5\ 1\ 3)(2\ 4)$ gives $h=(1\ 5\ 4\ 2)$,
$(5\ 1\ 3)(4\ 2)$ gives $h=(1\ 5\ 2)$.

The remaining ways of writing $(1\ 2)(3\ 4\ 5)$ give the following conjugating elements g.

$(3\ 4\ 5)(2\ 1)$ gives $g=(1\ 3\ 4)$,
$(4\ 5\ 3)(1\ 2)$ gives $g=(1\ 4\ 2)(3\ 5)$,
$(4\ 5\ 3)(2\ 1)$ gives $g=(1\ 4)(3\ 5)$,
$(5\ 3\ 4)(1\ 2)$ gives $g=(1\ 5\ 4\ 2)$,
$(5\ 3\ 4)(2\ 1)$ gives $g=(1\ 5\ 4)$.)

Solution to Additional Exercise E24

We can start by working out the orders of the elements of G. These are as follows.

Element	e	a	b	c	w	x	y	z
Order	1	4	2	4	4	4	4	4

(Unfortunately this is not very helpful for this group, as all but two elements have the same order 4.)

The partition of G by the orders of its elements is

$$\{e\}, \quad \{b\}, \quad \{a, c, w, x, y, z\}.$$

Thus the sets $\{e\}$ and $\{b\}$ are conjugacy classes.

Consider the set $\{a, c, w, x, y, z\}$. Conjugating a by e or a gives a, and

$$\begin{aligned}
bab^{-1} &= b(ab) = bc = a, \\
cac^{-1} &= c(aa) = cb = a, \\
waw^{-1} &= w(ay) = wz = c, \\
xax^{-1} &= x(az) = xw = c, \\
yay^{-1} &= y(aw) = yx = c \\
zaz^{-1} &= z(ax) = zy = c.
\end{aligned}$$

Hence $\{a, c\}$ is a conjugacy class.

Also, conjugating w by e or w gives w, and

$$\begin{aligned}
awa^{-1} &= a(wc) = ax = y, \\
bwb^{-1} &= b(wb) = by = w, \\
cwc^{-1} &= c(wa) = cz = y, \\
xwx^{-1} &= x(wz) = xc = y, \\
ywy^{-1} &= y(ww) = yb = w \\
zwz^{-1} &= z(wx) = za = y.
\end{aligned}$$

Hence $\{w, y\}$ is a conjugacy class.

Finally, we have

$$cxc^{-1} = c(xa) = cw = z.$$

Hence $\{x, z\}$ is a conjugacy class.

In summary, the conjugacy classes of G are

$$\{e\}, \quad \{b\}, \quad \{a, c\}, \quad \{w, y\}, \quad \{x, z\}.$$

Solution to Additional Exercise E25

We show that the three subgroup properties hold for $C(x)$.

SG1 Let $g, h \in C(x)$. Then $gxg^{-1} = x$ and $hxh^{-1} = x$. To show that $gh \in C(x)$, we have to show that

$$(gh)x(gh)^{-1} = x.$$

Now

$$\begin{aligned}
(gh)x(gh)^{-1} &= (gh)x(h^{-1}g^{-1}) \\
&= g(hxh^{-1})g^{-1} \\
&= gxg^{-1} \quad (\text{since } h \in C(x)) \\
&= x \quad (\text{since } g \in C(x)).
\end{aligned}$$

So $gh \in C(x)$. Thus property SG1 holds.

SG2 We have $exe^{-1} = x$, so $e \in C(x)$. Thus property SG2 holds.

SG3 Let $g \in C(x)$. Then

$$gxg^{-1} = x.$$

To show that $g^{-1} \in C(x)$, we have to show that

$$g^{-1}x(g^{-1})^{-1} = x.$$

Composing each side of the equation $gxg^{-1} = x$ on the left by g^{-1} and on the right by g, we obtain

$$g^{-1}(gxg^{-1})g = g^{-1}xg,$$

that is

$$x = g^{-1}xg,$$

which can be written as

$$g^{-1}x(g^{-1})^{-1} = x.$$

So $g^{-1} \in C(x)$. Thus property SG3 holds.

Hence $C(x)$ satisfies the three subgroup properties and so is a subgroup of G.

(The subset $C(x)$ can also be written as

$$C(x) = \{g \in G : gx = xg\}.$$

If x and y are elements of a group G, then we say that x *commutes* with y if $xy = yx$. So $C(x)$ is the set of all elements of G that commute with x. The set $C(x)$ is called the *centraliser* of x.)

Solution to Additional Exercise E26

(a) The group $(\mathbb{R}, +)$ is abelian, so each conjugacy class consists of a single element. In particular, π is conjugate only to itself, so its conjugacy class is $\{\pi\}$.

(b) The group $(\mathbb{R}^*, \times)$ is also abelian, so again the conjugacy class of π is $\{\pi\}$.

(There is nothing special about the number π or the particular numerical groups here: every number in an abelian group of numbers lies in a single-element conjugacy class.)

Solution to Additional Exercise E27

In $S(\square)$, the reflection r is conjugate only to itself and the reflection t. (This was established in Worked Exercise E28 in Subsection 2.3 of Unit E2.) Hence, for all $g \in S(\square)$,

$$\begin{aligned} g\{e, r\}g^{-1} &= \{g \circ e \circ g^{-1}, g \circ r \circ g^{-1}\} \\ &= \{e, r\} \text{ or } \{e, t\}. \end{aligned}$$

Thus the subgroup $\{e, r\}$ is conjugate only to itself and the subgroup $\{e, t\}$.

Solution to Additional Exercise E28

(a) First we show that $gHg^{-1} \subseteq \langle ghg^{-1} \rangle$. To do this, we have to show that every element of gHg^{-1} can be expressed as a power of ghg^{-1}. Since each element of H is of the form h^r for some integer r, each element of gHg^{-1} is of the form gh^rg^{-1} for some integer r. But, by Lemma E20 in Unit E2,

$$gh^rg^{-1} = (ghg^{-1})^r.$$

So each element of gHg^{-1} can be expressed as a power of ghg^{-1}. Thus $gHg^{-1} \subseteq \langle ghg^{-1} \rangle$.

Now we show that $\langle ghg^{-1} \rangle \subseteq gHg^{-1}$. The element ghg^{-1} is in gHg^{-1}, by the definition of gHg^{-1}. Hence, since gHg^{-1} is a subgroup (by Theorem E29 in Unit E2), all the powers of ghg^{-1} are in gHg^{-1}; that is, $\langle ghg^{-1} \rangle \subseteq gHg^{-1}$.

Therefore $gHg^{-1} = \langle ghg^{-1} \rangle$.

(b) The subgroup

$$\{e, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\}$$

of S_4 is cyclic, generated by $(1\ 2\ 3\ 4)$.

By part (a), if we conjugate this subgroup by any element $g \in S_4$, then the resulting subgroup is cyclic, generated by $g \circ (1\ 2\ 3\ 4) \circ g^{-1}$.

The conjugacy class of $(1\ 2\ 3\ 4)$ is the set of all 4-cycles in S_4. Thus the given subgroup is conjugate to each of the cyclic subgroups of S_4 generated by a 4-cycle. There are three such cyclic subgroups, namely

$$\{e, (1\ 2\ 3\ 4), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)\},$$
$$\{e, (1\ 2\ 4\ 3), (1\ 4)(2\ 3), (1\ 3\ 4\ 2)\},$$
$$\{e, (1\ 3\ 2\ 4), (1\ 2)(3\ 4), (1\ 4\ 2\ 3)\}.$$

Solution to Additional Exercise E29

(a) We use Theorem E33 (Property B) from Unit E2.

We have, for example, $(2, 0) \in C$ and $(1, 1) \in X$. The conjugate of $(2, 0)$ by $(1, 1)$ is

$$\begin{aligned} (1, 1) * (2, 0) * (1, 1)^{-1} &= (2, 1) * (1, -1) \\ &= (2, -1). \end{aligned}$$

Now $(2, -1) \notin C$ because its second coordinate is non-zero. It follows by Theorem E33 that C is not a normal subgroup of X.

(b) **(i)** In the group X we have

$$(1, 2)^{-1} = (1/1, -2/1) = (1, -2)$$

so

$$\begin{aligned} &(1, 2)C(1, 2)^{-1} \\ &= \{(1, 2) * (a, 0) * (1, 2)^{-1} : (a, 0) \in C\} \\ &= \{(a, 2) * (1, -2) : a \in \mathbb{R}^*\} \\ &= \{(a, -2a + 2) : a \in \mathbb{R}^*\}. \end{aligned}$$

(ii) In the group X we have

$$(2, 1)^{-1} = \left(\tfrac{1}{2}, -\tfrac{1}{2}\right)$$

so

$$\begin{aligned} &(2, 1)C(2, 1)^{-1} \\ &= \{(2, 1) * (a, 0) * (2, 1)^{-1} : (a, 0) \in C\} \\ &= \{(2a, 1) * (\tfrac{1}{2}, -\tfrac{1}{2}) : a \in \mathbb{R}^*\} \\ &= \{(a, -a + 1) : a \in \mathbb{R}^*\}. \end{aligned}$$

(Each of these sets is a subgroup of X, by Theorem E29 in Unit E2.)

Solution to Additional Exercise E30

(a) We use Theorem E33 (Property B) from Unit E2.

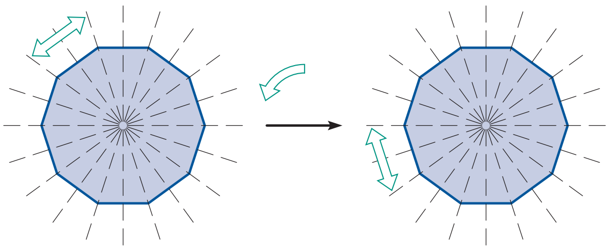
Let $x \in H \cap N$ and $h \in H$. Since $x \in N$, $h \in G$ and N is a normal subgroup of G, we have $hxh^{-1} \in N$. Also, since $x, h \in H$ and H is a subgroup we have $hxh^{-1} \in H$. Hence $hxh^{-1} \in H \cap N$. Thus, by Theorem E33, $H \cap N$ is a normal subgroup of H.

(b) Here is one example. Let $G = S(\square)$, let $N = \{e, b, r, t\}$ and let $H = \{e, r\}$. Then N is a subgroup of G (see Exercise E15 in Subsection 1.4 of Unit E1) and $H = \langle r \rangle$ is also a subgroup of G. Also N is normal in G since it has index 2 in G. However, $H \cap N = \{e, r\} = N$ is not normal in G, by the solution to Worked Exercise E31 in Subsection 3.3 of Unit E2.

Solution to Additional Exercise E31

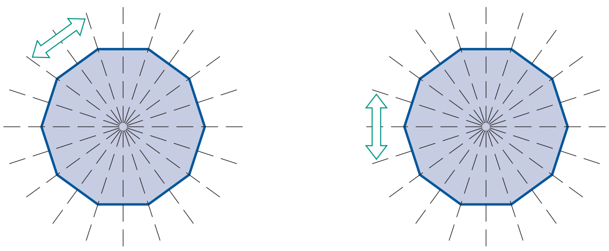
(a) These symmetries are conjugate in $S(\text{decagon})$.

A conjugating symmetry is the rotation through $2\pi/5$ anticlockwise (or the reflection in the axis midway between the axes of the given reflections).



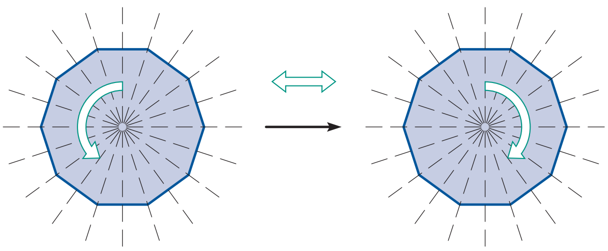
(b) These symmetries are not conjugate in $S(\text{decagon})$.

(There is no symmetry of the decagon that transforms a diagram illustrating the first of these symmetries into a diagram illustrating the second. The first symmetry is a reflection in an axis through midpoints of opposite sides, whereas the second symmetry is a reflection in an axis through opposite vertices.)



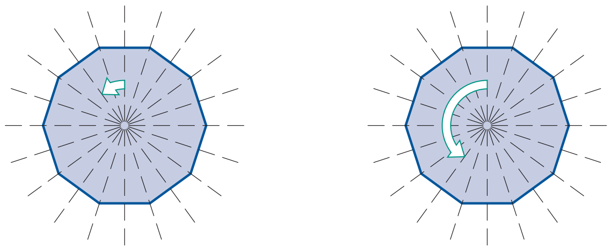
(c) These symmetries are conjugate in $S(\text{decagon})$.

A conjugating symmetry is the reflection in the vertical axis (or any reflection).



(d) These symmetries are not conjugate in $S(\text{decagon})$.

(There is no symmetry of the decagon that transforms a diagram illustrating the first of these symmetries into a diagram illustrating the second.)



Solution to Additional Exercise E32

(a) The conjugacy classes of the regular decagon are as follows.

- A class containing the identity symmetry alone.
- A class containing the rotation through π alone.
- Four classes each containing two rotations, namely the rotations through $k\pi/5$ and $-k\pi/5$ for $k = 1, 2, 3, 4$.
- A class containing all the reflections in axes through midpoints of opposite sides.
- A class containing all the reflections in axes through opposite vertices.

(Thus the symmetry group of the regular decagon has eight conjugacy classes altogether.)

(b) The conjugacy classes of the regular nonagon are as follows.

- A class containing the identity symmetry alone.
- Four classes each containing two rotations, namely the rotations through $2k\pi/9$ and $-2k\pi/9$ for $k = 1, 2, 3, 4$.
- A class containing all the reflections.

(Thus the symmetry group of the regular nonagon has six conjugacy classes altogether.)

Solution to Additional Exercise E33

(a) The elements of $S^+(P)$ are as follows.

e	
$(3\ 4\ 5\ 6)$	rotation through $\pi/2$ about axis A
$(3\ 5)(4\ 6)$	rotation through π about axis A
$(3\ 6\ 5\ 4)$	rotation through $3\pi/2$ about axis A
$(1\ 2)(3\ 5)$	rotation through π about axis B
$(1\ 2)(4\ 6)$	rotation through π about axis C
$(1\ 2)(3\ 4)(5\ 6)$	rotation through π about axis D
$(1\ 2)(3\ 6)(4\ 5)$	rotation through π about axis E

(b) (Note that Strategy E6 from Unit E2 does not apply directly here since the group under consideration is not the whole symmetry group, but we can use ideas from Strategy E6.)

The partition of $S^+(P)$ by cycle structure is as follows.

$\{e\}$

$\{(3\ 4\ 5\ 6), (3\ 6\ 5\ 4)\}$

$\{(3\ 5)(4\ 6), (1\ 2)(3\ 5), (1\ 2)(4\ 6)\}$

$\{(1\ 2)(3\ 4)(5\ 6), (1\ 2)(3\ 6)(4\ 5)\}$

One conjugacy class is $\{e\}$.

Now consider the cycle structure class

$$\{(3\ 4\ 5\ 6), (3\ 6\ 5\ 4)\}.$$

None of the first four elements of $S^+(P)$ listed in the solution to part (a) above will conjugate $(3\ 4\ 5\ 6)$ to $(3\ 6\ 5\ 4)$, since they are all elements of the abelian subgroup $\langle(3\ 4\ 5\ 6)\rangle$ generated by $(3\ 4\ 5\ 6)$. Let us try the element $(1\ 2)(3\ 4)(5\ 6)$ of $S^+(P)$:

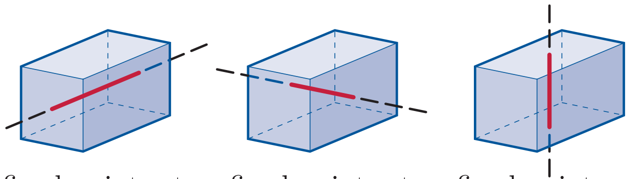
$$\begin{array}{ll} & (3\ 4\ 5\ 6) \\ (1\ 2)(3\ 4)(5\ 6) & \ \downarrow\ \downarrow\ \downarrow\ \downarrow \\ & (4\ 3\ 6\ 5) = (3\ 6\ 5\ 4). \end{array}$$

Since this conjugates $(3\ 4\ 5\ 6)$ to $(3\ 6\ 5\ 4)$, the cycle structure class above is a conjugacy class.

Now consider the cycle structure class

$$\{(3\ 5)(4\ 6), (1\ 2)(3\ 5), (1\ 2)(4\ 6)\}.$$

There is no symmetry of the prism, and hence in particular no direct symmetry of the prism, that maps the fixed point set of $(3\ 5)(4\ 6)$ to the fixed point set of $(1\ 2)(3\ 5)$ or to the fixed point set of $(1\ 2)(4\ 6)$ (see the figures below), so $(3\ 5)(4\ 6)$ is not conjugate to either $(1\ 2)(3\ 5)$ or $(1\ 2)(4\ 6)$ in $S^+(P)$.



fixed point set of $(3\ 5)(4\ 6)$ | fixed point set of $(1\ 2)(3\ 5)$ | fixed point set of $(1\ 2)(4\ 6)$

We now have to determine whether $(1\ 2)(3\ 5)$ is conjugate to $(1\ 2)(4\ 6)$ in $S^+(P)$. Let us try conjugating $(1\ 2)(3\ 5)$ by the element $(3\ 4\ 5\ 6)$ of $S^+(P)$:

$$\begin{array}{ll} & (1\ 2)(3\ 5) \\ (3\ 4\ 5\ 6) & \ \downarrow\downarrow\ \ \downarrow\downarrow \\ & (1\ 2)(4\ 6). \end{array}$$

Since this conjugates $(1\ 2)(3\ 5)$ to $(1\ 2)(4\ 6)$, we have now found that the cycle structure class above splits into two conjugacy classes:

$$\{(3\ 5)(4\ 6)\}, \quad \{(1\ 2)(3\ 5), (1\ 2)(4\ 6)\}.$$

Finally consider the cycle structure class

$$\{(1\ 2)(3\ 4)(5\ 6), (1\ 2)(3\ 6)(4\ 5)\}.$$

Let us try conjugating $(1\ 2)(3\ 4)(5\ 6)$ by the element $(3\ 4\ 5\ 6)$ of $S^+(P)$:

$$\begin{array}{ll} & (1\ 2)(3\ 4)(5\ 6) \\ (3\ 4\ 5\ 6) & \ \downarrow\downarrow\ \ \downarrow\downarrow\ \ \downarrow\downarrow \\ & (1\ 2)(4\ 5)(6\ 3) = (1\ 2)(3\ 6)(4\ 5). \end{array}$$

Since this conjugates $(1\ 2)(3\ 4)(5\ 6)$ to $(1\ 2)(3\ 6)(4\ 5)$, the cycle structure class above is a conjugacy class.

In summary, the conjugacy classes of $S^+(P)$ are as follows.

$\{e\}$

$\{(3\ 4\ 5\ 6), (3\ 6\ 5\ 4)\}$

$\{(3\ 5)(4\ 6)\}$

$\{(1\ 2)(3\ 5), (1\ 2)(4\ 6)\}$

$\{(1\ 2)(3\ 4)(5\ 6), (1\ 2)(3\ 6)(4\ 5)\}$

(Notice that the group $S^+(P)$ has order 8 and is non-abelian with exactly six self-inverse elements, so it is isomorphic to $S(\square)$. Any isomorphism from $S(\square)$ to $S^+(P)$ will convert the conjugacy classes of $S(\square)$ to those of $S^+(P)$. In particular, $S^+(P)$ must have conjugacy classes of the same sizes as those of $S(\square)$, namely 1, 1, 2, 2 and 2.)

Solution to Additional Exercise E34

(a) By the solution to Exercise E93 in Subsection 4.2 of Unit E2 (or by the method of Additional Exercise E32), the conjugacy classes of $S(\hexagon)$ are as follows. They have been labelled $A, B, \ldots, F$.

$$A = \{e\}$$
$$B = \{a_1, a_5\}$$
$$C = \{a_2, a_4\}$$
$$D = \{a_3\}$$
$$E = \{t_1, t_3, t_5\}$$
$$F = \{t_2, t_4, t_6\}$$

(b) We apply Strategy E5.

The group $S(\hexagon)$ has six conjugacy classes, and the numbers of elements in these classes are 1, 1, 2, 2, 3, 3.

We need to find all the unions of conjugacy classes that include the class $A = \{e\}$ and whose total number of elements is a divisor of $|S(\hexagon)| = 12$.

So we seek ways of adding some of the numbers

$$1, 1, 2, 2, 3, 3,$$

always including 1, to obtain a total that is one of the numbers 1, 2, 3, 4, 6 or 12.

There are eight suitable sums of numbers:

$$1, \quad 1+1=2, \quad 1+2=3,$$
$$1+1+2=4, \quad 1+3=4,$$
$$1+1+2+2=6, \quad 1+2+3=6,$$
$$1+1+2+2+3+3=12.$$

Thus the only unions of conjugacy classes that include $A = \{e\}$ and have a permissible number of elements are as follows:

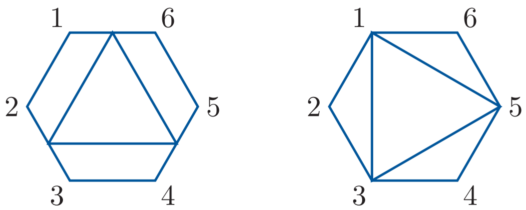
1. $A = \{e\}$
2. $A \cup D = \{e, a_3\}$
3. $A \cup B = \{e, a_1, a_5\}$
4. $A \cup C = \{e, a_2, a_4\}$
5. $A \cup D \cup B = \{e, a_1, a_3, a_5\}$
6. $A \cup D \cup C = \{e, a_2, a_3, a_4\}$
7. $A \cup E = \{e, t_1, t_3, t_5\}$
8. $A \cup F = \{e, t_2, t_4, t_6\}$
9. $A \cup D \cup B \cup C = \{e, a_1, a_2, a_3, a_4, a_5\}$
10. $A \cup B \cup E = \{e, a_1, a_5, t_1, t_3, t_5\}$
11. $A \cup B \cup F = \{e, a_1, a_5, t_2, t_4, t_6\}$
12. $A \cup C \cup E = \{e, a_2, a_4, t_1, t_3, t_5\}$
13. $A \cup C \cup F = \{e, a_2, a_4, t_2, t_4, t_6\}$
14. $A \cup B \cup C \cup D \cup E \cup F = S(\hexagon)$

If any of these sets is a subgroup, then it is a normal subgroup, by Theorem E32 in Unit E2.

The first and last sets are the trivial subgroup $\{e\}$ and the whole group $S(\hexagon)$ respectively.

Sets 2, 4 and 9 are the cyclic subgroups generated by a_3, a_2 and a_1, respectively.

Sets 12 and 13 are the symmetry groups of the following two modified hexagons, respectively.



(An alternative way to check that each of sets 12 and 13 is a subgroup of $S(\hexagon)$ is to construct a Cayley table and verify the three subgroup properties.)

So all these sets are subgroups and hence normal subgroups of $S(\hexagon)$.

None of the other sets in the list above is a subgroup of $S(\hexagon)$. We can confirm this by showing that the closure axiom, G1, fails in each case, as follows. For sets 3, 5, 10 and 11, the element a_1 is in the set but $a_1 \circ a_1 = a_2$ is not. For set 6, the elements a_2 and a_3 are in the set but $a_2 \circ a_3 = a_5$ is not. For set 7, the composite $t_1 \circ t_3$ of two different reflections must be a non-trivial rotation, but there is no such element in the set. Similarly, for set 8, the composite $t_2 \circ t_4$ of two different reflections must be a non-trivial rotation, but there is no such element in the set.

In summary, the group $S(\hexagon)$ has seven normal subgroups, as follows.

$$\{e\}$$
$$\{e, a_3\}$$
$$\{e, a_2, a_4\}$$
$$\{e, a_1, a_2, a_3, a_4, a_5\}$$
$$\{e, a_2, a_4, t_1, t_3, t_5\}$$
$$\{e, a_2, a_4, t_2, t_4, t_6\}$$
$$S(\hexagon)$$

Solution to Additional Exercise E35

We have

$$\begin{aligned}
&\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} D \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{-1}\\
&= \left\{\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{-1} : \begin{pmatrix}a & 0\\ 0 & d\end{pmatrix} \in D\right\}\\
&= \left\{\begin{pmatrix}a & 0\\ a & d\end{pmatrix} \times \frac{1}{1}\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix} : a, d \in \mathbb{R},\ ad \neq 0\right\}\\
&= \left\{\begin{pmatrix}a & 0\\ a-d & d\end{pmatrix} : a, d \in \mathbb{R},\ ad \neq 0\right\}.
\end{aligned}$$

This subgroup is not equal to L because, for example,

$$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} \in L,$$

since this matrix is lower triangular and invertible, but

$$\begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix} \notin \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix} D \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^{-1},$$

because there are no numbers $a, d \in \mathbb{R}$ such that

$$\begin{pmatrix}a & 0\\ a-d & d\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 2\end{pmatrix}.$$

This is because if $a = 1$ and $d = 2$ then

$$a - d = -1 \neq 0.$$

Solution to Additional Exercise E36

(a) We have

$$\begin{aligned}
&\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} H \begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{-1}\\
&= \left\{\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1/a\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}^{-1} : \begin{pmatrix}a & 0\\ 0 & 1/a\end{pmatrix} \in H\right\}\\
&= \left\{\begin{pmatrix}a & 1/a\\ 0 & 1/a\end{pmatrix}\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix} : a \in \mathbb{R}^*\right\}\\
&= \left\{\begin{pmatrix}a & (1/a)-a\\ 0 & 1/a\end{pmatrix} : a \in \mathbb{R}^*\right\}\\
&= \left\{\begin{pmatrix}a & (1-a^2)/a\\ 0 & 1/a\end{pmatrix} : a \in \mathbb{R}^*\right\}.
\end{aligned}$$

(By taking $b = 1/a$, we can alternatively specify this subgroup as

$$\left\{\begin{pmatrix}a & b-a\\ 0 & b\end{pmatrix} : a, b \in \mathbb{R}^*,\ ab = 1\right\}.$$

Either specification is fine: in the first the matrix is more complicated but the condition is simpler; in the second the matrix is simpler but the conditions are more complicated.)

(b) We have

$$\begin{aligned}
&\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix} H \begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}^{-1}\\
&= \left\{\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & 1/a\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 2\end{pmatrix}^{-1} : \begin{pmatrix}a & 0\\ 0 & 1/a\end{pmatrix} \in H\right\}\\
&= \left\{\begin{pmatrix}0 & 1/a\\ a & 2/a\end{pmatrix}\begin{pmatrix}-2 & 1\\ 1 & 0\end{pmatrix} : a \in \mathbb{R}^*\right\}\\
&= \left\{\begin{pmatrix}1/a & 0\\ (2/a)-2a & a\end{pmatrix} : a \in \mathbb{R}^*\right\}\\
&= \left\{\begin{pmatrix}1/a & 0\\ 2(1-a^2)/a & a\end{pmatrix} : a \in \mathbb{R}^*\right\}.
\end{aligned}$$

(By taking $b = 1/a$, we can alternatively specify this subgroup as

$$\left\{\begin{pmatrix}b & 0\\ 2(b-a) & a\end{pmatrix} : a, b \in \mathbb{R}^*,\ ab = 1\right\}.$$

Either specification is fine: as in part (a), in the first the matrix is more complicated but the condition is simpler, whereas in the second the matrix is simpler but the conditions are more complicated.)

Solution to Additional Exercise E37

(a) We have

$$\begin{aligned}
&\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} V \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^{-1}\\
&= \left\{ \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \begin{pmatrix}1+n & n\\ -n & 1-n\end{pmatrix} \begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^{-1} : \begin{pmatrix}1+n & n\\ -n & 1-n\end{pmatrix} \in V \right\}\\
&= \left\{ \begin{pmatrix}1-n & 2-n\\ -n & 1-n\end{pmatrix} \begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix} : n \in \mathbb{Z} \right\}\\
&= \left\{ \begin{pmatrix}1-n & n\\ -n & 1+n\end{pmatrix} : n \in \mathbb{Z} \right\}.
\end{aligned}$$

This subgroup is not equal to V because, for example, the matrix

$$\begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix},$$

is in V because it is obtained by taking $n = 1$ in the matrix in the definition of V, but it is not in the subgroup $\mathbf{B}V\mathbf{B}^{-1}$ here because there is no integer n such that

$$\begin{pmatrix}1-n & n\\ -n & 1+n\end{pmatrix} = \begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix}.$$

This is because if $n = 1$ (as required by the top right entry) then

$$1 - n = 0 \neq 2.$$

(b) We have

$$\begin{aligned}
&\begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix} V \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}^{-1}\\
&= \left\{ \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix} \begin{pmatrix}1+n & n\\ -n & 1-n\end{pmatrix} \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}^{-1} : \begin{pmatrix}1+n & n\\ -n & 1-n\end{pmatrix} \in V \right\}\\
&= \left\{ \begin{pmatrix}1+n & n\\ 3+2n & 1+2n\end{pmatrix} \begin{pmatrix}1 & 0\\ -3 & 1\end{pmatrix} : n \in \mathbb{Z} \right\}\\
&= \left\{ \begin{pmatrix}1-2n & n\\ -4n & 1+2n\end{pmatrix} : n \in \mathbb{Z} \right\}
\end{aligned}$$

We can show that this subgroup is not equal to V by using a similar argument as in part (a): the matrix

$$\begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix},$$

is in V because it is obtained by taking $n = 1$ in the matrix in the definition of V, but it is not in the subgroup $\mathbf{B}V\mathbf{B}^{-1}$ here because there is no integer n such that

$$\begin{pmatrix}1-2n & n\\ -4n & 1+2n\end{pmatrix} = \begin{pmatrix}2 & 1\\ -1 & 0\end{pmatrix}.$$

This is because if $n = 1$ (as required by the top right entry) then

$$-4n = -4 \neq -1.$$

Solution to Additional Exercise E38

The subgroup R contains only lower triangular matrices and hence it is a subset of L. Since it is a subgroup of $\mathrm{GL}(2)$, it follows that it is a subgroup of L.

To show that R is a normal subgroup of L, we use Property B of Theorem E33 from Unit E2.

We have to show that for every matrix $\mathbf{A} \in R$ and every matrix $\mathbf{B} \in L$, we have $\mathbf{BAB}^{-1} \in R$. Let $\mathbf{A} \in R$ and let $\mathbf{B} \in L$. Then

$$\mathbf{A} = \begin{pmatrix}x & 0\\ y & x\end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix}r & 0\\ t & u\end{pmatrix}$$

for some $x, y, r, t, u \in \mathbb{R}$ where $x \neq 0$ and $ru \neq 0$.

We have

$$\begin{aligned}
\mathbf{BAB}^{-1} &= \begin{pmatrix}r & 0\\ t & u\end{pmatrix} \begin{pmatrix}x & 0\\ y & x\end{pmatrix} \begin{pmatrix}r & 0\\ t & u\end{pmatrix}^{-1}\\
&= \begin{pmatrix}rx & 0\\ tx+uy & ux\end{pmatrix} \times \frac{1}{ru} \begin{pmatrix}u & 0\\ -t & r\end{pmatrix}\\
&= \frac{1}{ru} \begin{pmatrix}rux & 0\\ tux + u^2y - tux & rux\end{pmatrix}\\
&= \begin{pmatrix}x & 0\\ uy/r & x\end{pmatrix}.
\end{aligned}$$

This matrix is of the form

$$\begin{pmatrix}a & 0\\ c & a\end{pmatrix}$$

with $a = x$ and $c = uy/r$. Also $x \neq 0$. Hence $\mathbf{BAB}^{-1} \in R$.

Thus R is a normal subgroup of L.

Solution to Additional Exercise E39

(a) Let $\mathbf{A}$ be any matrix in U. Then

$$\mathbf{A} = \begin{pmatrix} p & q \\ 0 & s \end{pmatrix}$$

for some $p, q, s \in \mathbb{R}$ with $ps \neq 0$. The coset of M in U containing $\mathbf{A}$ is

$$\begin{aligned} \mathbf{A}M &= \left\{ \begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \mathbf{X} : \mathbf{X} \in M \right\} \\ &= \left\{ \begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\} \\ &= \left\{ \begin{pmatrix} pa & pb + qa \\ 0 & sa \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}. \end{aligned}$$

We need to find a matrix in this coset of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$$

where $x \in \mathbb{R}^*$. So we need to find values of $a, b \in \mathbb{R}$ with $a \neq 0$ such that

$$\begin{pmatrix} pa & pb + qa \\ 0 & sa \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$$

for some $x \in \mathbb{R}^*$.

Let $a = 1/p$ and $b = -q/p^2$ (these values are chosen so as to obtain 1 as the top left entry and 0 as the top right entry in the matrix). These values of a and b do exist since $p \neq 0$, they do satisfy $a, b \in \mathbb{R}$ with $a \neq 0$, and they give

$$\begin{aligned} \begin{pmatrix} pa & pb + qa \\ 0 & sa \end{pmatrix} &= \begin{pmatrix} p(1/p) & p(-q/p^2) + q(1/p) \\ 0 & s(1/p) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & s/p \end{pmatrix}. \end{aligned}$$

Now $s/p \neq 0$ since $s \neq 0$, so this is a matrix of the required form in the coset $\mathbf{A}M$.

Therefore

$$\begin{pmatrix} 1 & 0 \\ 0 & s/p \end{pmatrix} M = \mathbf{A}M.$$

Thus we have shown that the coset $\mathbf{A}M$ can be expressed as

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M$$

for some $x \in \mathbb{R}^*$. Since $\mathbf{A}$ is any matrix in U, this proves the required result.

(b) The solution to part (a) shows that every coset of M in U *can* be expressed as

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M$$

for some $x \in \mathbb{R}^*$. To show that there is a *unique* such value of x for each coset, we have to prove that if x and y are elements of $\mathbb{R}^*$ such that

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M$$

then $x = y$. So suppose that $x, y \in \mathbb{R}^*$ and the equation above holds. It follows that

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M.$$

Now

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \mathbf{X} : \mathbf{X} \in M \right\} \\ &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\} \\ &= \left\{ \begin{pmatrix} a & b \\ 0 & ay \end{pmatrix} : a, b \in \mathbb{R},\ a \neq 0 \right\}. \end{aligned}$$

Therefore

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & ay \end{pmatrix}$$

for some $a, b \in \mathbb{R}$ with $a \neq 0$. This equation gives $a = 1$ and hence $x = y$. This completes the required proof.

(c) Let the mapping ϕ be defined by

$$\begin{aligned} \phi : \mathbb{R}^* &\longrightarrow U/M \\ x &\longmapsto \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M. \end{aligned}$$

We show that ϕ is an isomorphism.

First we show that ϕ is one-to-one. Let $x, y \in \mathbb{R}^*$, and suppose that

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M.$$

Then by part (b) it follows that $x = y$. Thus ϕ is one-to-one.

Next we show that ϕ is onto. Consider any element of U/M. By part (a) it can be written as

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M$$

for some $x \in \mathbb{R}^*$. This coset is the image under ϕ of the element $x \in \mathbb{R}^*$. Thus ϕ is onto.

Finally we show that

$$\phi(x \times y) = \phi(x) \cdot \phi(y)$$

for all $x, y \in \mathbb{R}^*$ (where $\cdot$ denotes set composition). Let $x, y \in \mathbb{R}^*$. Then

$$\phi(x \times y) = \phi(xy) = \begin{pmatrix} 1 & 0 \\ 0 & xy \end{pmatrix} M$$

and

$$\begin{aligned} \phi(x) \cdot \phi(y) &= \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M \cdot \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M \\ &= \left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \right) M \\ &\quad \text{(by the rule for combining cosets of } M \text{ in } U) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & xy \end{pmatrix} M. \end{aligned}$$

Hence

$$\phi(x \times y) = \phi(x) \cdot \phi(y)$$

as required.

We have now shown that ϕ is an isomorphism. It follows that $U/M \cong (\mathbb{R}^*, \times)$, as required.

(Here is an alternative solution to part (c), involving an isomorphism from U/M to $(\mathbb{R}^*, \times)$ rather than from $(\mathbb{R}^*, \times)$ to U/M. It is a little more complicated to prove that $U/M \cong (\mathbb{R}^*, \times)$ in this way.

Let the mapping ϕ be defined by

$$\begin{aligned} \phi : U/M &\longrightarrow \mathbb{R}^* \\ \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M &\longmapsto x. \end{aligned}$$

This definition *does* specify a mapping (every element of the domain is mapped to exactly one element of the codomain) because, by part (b), every element of U/M can be expressed as

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M.$$

for a unique value of $x \in \mathbb{R}^*$.

We show that ϕ is an isomorphism.

First we show that ϕ is one-to-one. Consider any two cosets in U/M. Then by part (b) there are unique values $x, y \in \mathbb{R}^*$ such that these cosets can be expressed as

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M.$$

Suppose that

$$\phi\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M \right) = \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M \right).$$

Then, by the definition of ϕ,

$$x = y.$$

So the two cosets are the same coset. Thus ϕ is one-to-one.

Next we show that ϕ is onto. Any element $x \in \mathbb{R}^*$ is the image under ϕ of the element

$$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M$$

of U/M. Thus ϕ is onto.

Finally we show that

$$\phi(X \cdot Y) = \phi(X) \times \phi(Y)$$

for all cosets $X, Y \in U/M$.

Let $X, Y \in U/M$. Then by part (b) there are unique values $x, y \in \mathbb{R}^*$ such that

$$X = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M \quad \text{and} \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M.$$

Hence

$$\begin{aligned} \phi(X \cdot Y) &= \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M \cdot \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M \right) \\ &= \phi\left(\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \right) M \right) \\ &\quad \text{(by the rule for combining cosets of } M \text{ in } U) \\ &= \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & xy \end{pmatrix} M \right) \\ &= xy \end{aligned}$$

and

$$\begin{aligned} &\phi(X) \times \phi(Y) \\ &= \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} M \right) \times \phi\left(\begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} M \right) \\ &= x \times y \\ &= xy. \end{aligned}$$

Hence

$$\phi(X \cdot Y) = \phi(X) \times \phi(Y)$$

as required.

We have now shown that ϕ is an isomorphism. It follows that $U/M \cong (\mathbb{R}^*, \times)$, as required.)

Solution to Additional Exercise E40

(a) Let $\mathbf{A}$ be any matrix in U. Then

$$\mathbf{A} = \begin{pmatrix} p & q \\ 0 & s \end{pmatrix}$$

for some $p, q, s \in \mathbb{R}$ with $ps \neq 0$. The coset of S in U containing $\mathbf{A}$ is

$$\begin{aligned} \mathbf{A}S &= \left\{ \begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \mathbf{X} : \mathbf{X} \in S \right\} \\ &= \left\{ \begin{pmatrix} p & q \\ 0 & s \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} p & pb+q \\ 0 & s \end{pmatrix} : b \in \mathbb{R} \right\}. \end{aligned}$$

Now if we take $b = -q/p$ (this value does exist since $p \neq 0$), then we have

$$\begin{aligned} \begin{pmatrix} p & pb+q \\ 0 & s \end{pmatrix} &= \begin{pmatrix} p & p(-q/p)+q \\ 0 & s \end{pmatrix} \\ &= \begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} \in \mathbf{A}S$$

and hence

$$\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} S = \mathbf{A}S.$$

Now $p, s \in \mathbb{R}^*$ since $ps \neq 0$. Thus we have shown that the coset $\mathbf{A}S$ can be expressed as

$$\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} S$$

for some $p, s \in \mathbb{R}^*$. Since $\mathbf{A}$ is any matrix in U, this proves the required result.

(b) The solution to part (a) shows that every coset of S in U *can* be expressed as

$$\begin{pmatrix} p & 0 \\ 0 & s \end{pmatrix} S$$

for some $p, s \in \mathbb{R}^*$. To show that there is a *unique* pair of real numbers p, s for each coset, we have to prove that if p_1, s_1, p_2 and s_2 are elements of $\mathbb{R}^*$ such that

$$\begin{pmatrix} p_1 & 0 \\ 0 & s_1 \end{pmatrix} S = \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix} S$$

then $p_1 = p_2$ and $s_1 = s_2$. So suppose that $p_1, s_1, p_2, s_2 \in \mathbb{R}^*$ and the equation above holds.

It follows that

$$\begin{pmatrix} p_1 & 0 \\ 0 & s_1 \end{pmatrix} \in \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix} S.$$

Now

$$\begin{aligned} \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix} S &= \left\{ \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix} \mathbf{X} : \mathbf{X} \in S \right\} \\ &= \left\{ \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{R} \right\} \\ &= \left\{ \begin{pmatrix} p_2 & p_2 b \\ 0 & s_2 \end{pmatrix} : b \in \mathbb{R} \right\}. \end{aligned}$$

Therefore

$$\begin{pmatrix} p_1 & 0 \\ 0 & s_1 \end{pmatrix} = \begin{pmatrix} p_2 & p_2 b \\ 0 & s_2 \end{pmatrix}$$

for some $b \in \mathbb{R}$. This equation gives $p_1 = p_2$ and $s_1 = s_2$ (and $b = 0$). This completes the required proof.

(c) From the formula for combining cosets using set composition, the missing matrix is the matrix product in U:

$$\begin{pmatrix} p_1 & 0 \\ 0 & s_1 \end{pmatrix} \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix} = \begin{pmatrix} p_1 p_2 & 0 \\ 0 & s_1 s_2 \end{pmatrix}.$$

Hence the formula is

$$\begin{pmatrix} p_1 & 0 \\ 0 & s_1 \end{pmatrix} S \cdot \begin{pmatrix} p_2 & 0 \\ 0 & s_2 \end{pmatrix} S = \begin{pmatrix} p_1 p_2 & 0 \\ 0 & s_1 s_2 \end{pmatrix} S.$$

Additional exercises for Unit E3

Section 1

Additional Exercise E41

Find an isomorphism from the cyclic group $(\mathbb{Z}_7^*, \times_7)$ to the cyclic group $(\mathbb{Z}_6, +_6)$.

Additional Exercise E42

Show that the groups $(U_9, \times_9)$ and $(U_{14}, \times_{14})$ are both cyclic groups of order 6, and find an isomorphism from $(U_9, \times_9)$ to $(U_{14}, \times_{14})$.

Additional Exercise E43

Show that the mapping

$$\begin{aligned} \phi : S(\triangle) &\longrightarrow S(\triangle) \\ e, a, b &\longmapsto e \\ r, s, t &\longmapsto r \end{aligned}$$

is a homomorphism.

Hint: Notice that ϕ maps direct symmetries to e and indirect symmetries to r.

Additional Exercise E44

Show that each of the following mappings ϕ is a homomorphism.

(a) $\begin{aligned} \phi : (\mathbb{R}^*, \times) &\longrightarrow (\mathbb{R}^*, \times) \\ x &\longmapsto x^4 \end{aligned}$

(b) $\begin{aligned} \phi : (\mathbb{R}^*, \times) &\longrightarrow (\mathbb{R}^*, \times) \\ x &\longmapsto x^5 \end{aligned}$

(c) $\begin{aligned} \phi : (\mathbb{C}^*, \times) &\longrightarrow (\mathbb{C}^*, \times) \\ z &\longmapsto \overline{z} \end{aligned}$

(d) $\begin{aligned} \phi : (\mathbb{R}^2, +) &\longrightarrow (\mathbb{R}, +) \\ (x, y) &\longmapsto x - 2y \end{aligned}$

Additional Exercise E45

Determine which of the homomorphisms in Additional Exercise E44 are isomorphisms.

Additional Exercise E46

Recall that (in this module) L denotes the group of invertible 2×2 lower triangular matrices with real entries under matrix multiplication:

$$L = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} : ad \neq 0 \right\}.$$

Show that the following mapping is a homomorphism.

$$\begin{aligned} \phi : (L, \times) &\longrightarrow (L, \times) \\ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \end{aligned}$$

Additional Exercise E47

Show that the following mapping is a homomorphism. Here $M_{2,2}$ is the group of all 2×2 matrices with real entries under matrix addition.

$$\begin{aligned} \phi : (M_{2,2}, +) &\longrightarrow (\mathbb{R}, +) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto a + d \end{aligned}$$

(It was mentioned at the start of Section 2 in Unit E1 that for any positive integers m and n the set $M_{m,n}$ of all $m \times n$ matrices with real entries is a group under matrix addition.)

Additional Exercise E48

Show that neither of the following mappings ϕ is a homomorphism.

(a) $\begin{aligned} \phi : (\mathbb{R}^2, +) &\longrightarrow (\mathbb{R}, +) \\ (x, y) &\longmapsto x + y + 1 \end{aligned}$

(b) $\begin{aligned} \phi : (\mathbb{R}, +) &\longrightarrow (\mathbb{R}, +) \\ x &\longmapsto x^2 \end{aligned}$

Additional Exercise E49

Let $(G, \circ)$ be an abelian group. Show that the mapping

$$\begin{aligned} \phi : (G, \circ) &\longrightarrow (G, \circ) \\ x &\longmapsto x^{-1} \end{aligned}$$

is an automorphism of $(G, \circ)$.

(The result in Exercise E103 in Subsection 1.1 of Unit E3 is a particular case of this result.)

Additional Exercise E50

Let $(G, \circ)$ be a group and let g be a particular element of G. Show that the mapping

$$\begin{aligned} \phi_g : (G, \circ) &\longrightarrow (G, \circ) \\ x &\longmapsto g \circ x \circ g^{-1} \end{aligned}$$

is an isomorphism.

(Isomorphisms obtained from conjugation in this way are important in the advanced study of groups.)

Additional Exercise E51 Challenging

Let $(G, *)$ be a group and let A be the set of all automorphisms of $(G, *)$. Prove that A is a group under function composition.

Section 2

Additional Exercise E52

Find the image and the kernel of each of the following homomorphisms.

(a) $$\begin{aligned} \phi : (\mathbb{R}^*, \times) &\longrightarrow (\mathbb{R}^*, \times) \\ x &\longmapsto x^4 \end{aligned}$$

(b) $$\begin{aligned} \phi : (\mathbb{R}^*, \times) &\longrightarrow (\mathbb{R}^*, \times) \\ x &\longmapsto x^5 \end{aligned}$$

(c) $$\begin{aligned} \phi : (\mathbb{C}^*, \times) &\longrightarrow (\mathbb{C}^*, \times) \\ z &\longmapsto \overline{z} \end{aligned}$$

(d) $$\begin{aligned} \phi : (\mathbb{R}^2, +) &\longrightarrow (\mathbb{R}, +) \\ (x, y) &\longmapsto x - 2y \end{aligned}$$

(These mappings are shown to be homomorphisms in the solution to Additional Exercise E44. You may find it helpful to refer to the solution to Additional Exercise E45.)

Additional Exercise E53

Find the image and the kernel of the homomorphism

$$\begin{aligned} \phi : (L, \times) &\longrightarrow (L, \times) \\ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \end{aligned}$$

where L is the group of invertible 2×2 lower triangular matrices.

(This mapping is shown to be a homomorphism in the solution to Additional Exercise E46.)

Additional Exercise E54

Show that the mapping

$$\begin{aligned} \phi : (L, \times) &\longrightarrow (\mathbb{R}^*, \times) \\ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} &\longmapsto a, \end{aligned}$$

where L is the group of invertible 2×2 lower triangular matrices, is a homomorphism, and find its image and kernel.

Additional Exercise E55

Find the image and the kernel of the homomorphism

$$\begin{aligned} \phi : (M_{2,2}, +) &\longrightarrow (\mathbb{R}, +) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\longmapsto a + d, \end{aligned}$$

where $(M_{2,2}, +)$ is the group of 2×2 matrices with real entries under matrix addition.

(This mapping is shown to be a homomorphism in the solution to Additional Exercise E47.)

Additional Exercise E56

Let M be the following set of invertible 2×2 matrices:

$$M = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 - b^2 \neq 0 \right\}.$$

Notice that

$$a^2 - b^2 = \det \begin{pmatrix} a & b \\ b & a \end{pmatrix}.$$

(a) Prove that $(M, \times)$ is a subgroup of $\mathrm{GL}(2)$.

(b) Show that the mapping

$$\begin{aligned} \phi : (M, \times) &\longrightarrow (\mathbb{R}^*, \times) \\ \begin{pmatrix} a & b \\ b & a \end{pmatrix} &\longmapsto a - b \end{aligned}$$

is a homomorphism.

(c) Show that ϕ is onto, and find its kernel.

Additional Exercise E57

Show that the mapping

$$\begin{aligned} \phi : (\mathbb{C}^*, \times) &\longrightarrow (\mathbb{C}^*, \times) \\ z &\longmapsto z/|z| \end{aligned}$$

is a homomorphism, and find its image and kernel.

Section 3

Additional Exercise E58

Consider the mapping

$$\begin{aligned} \phi : (\mathbb{C}^*, \times) &\longrightarrow (\mathbb{C}^*, \times) \\ z &\longmapsto z^4. \end{aligned}$$

(a) Show that ϕ is a homomorphism.

(b) Find $\operatorname{Im}\phi$ and $\operatorname{Ker}\phi$.

(c) Find a standard group isomorphic to the quotient group $\mathbb{C}^*/\operatorname{Ker}\phi$.

Additional Exercise E59

The homomorphism

$$\begin{aligned} \phi : (L, \times) &\longrightarrow (L, \times) \\ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} &\longmapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}, \end{aligned}$$

where L is the group of all invertible 2×2 lower triangular matrices, has image

$$\operatorname{Im}\phi = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in \mathbb{R}^* \right\}$$

and kernel

$$\operatorname{Ker}\phi = \left\{ \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} : c, d \in \mathbb{R},\ d \neq 0 \right\}.$$

(This mapping was shown to be a homomorphism in the solution to Additional Exercise E46, and its image and kernel were found in the solution to Additional Exercise E53.)

Find a standard group isomorphic to the quotient group $L/\operatorname{Ker}\phi$.

Additional Exercise E60 Challenging

The homomorphism

$$\begin{aligned} \phi : (\mathbb{C}^*, \times) &\longrightarrow (\mathbb{C}^*, \times) \\ z &\longmapsto z/|z| \end{aligned}$$

has image

$$\operatorname{Im}\phi = \{z \in \mathbb{C} : |z| = 1\}$$

and kernel

$$\operatorname{Ker}\phi = \mathbb{R}^+.$$

Show that the quotient group $\mathbb{C}^*/\operatorname{Ker}\phi$ is isomorphic to the group $((-\pi, \pi], +_{2\pi})$.

(The mapping ϕ was shown to be a homomorphism, and its image and kernel were found, in the solution to Additional Exercise E57.

The operation $+_{2\pi}$ is defined on the interval $(-\pi, \pi]$. This operation, and its application to principal arguments of complex numbers, are discussed near the end of Subsection 4.2 of Unit A3. You may assume here that $((-\pi, \pi], +_{2\pi})$ is a group. It is straightforward to prove the closure, identity and inverses axioms, and the associativity of the operation $+_{2\pi}$ follows from the associativity of the operation $+$ on $\mathbb{R}$.)

Note: The next three exercises are based on the optional *material in Subsection 3.3 of Unit E3. You should attempt them only if you have studied this optional material and are interested in trying further optional exercises on this topic. There will be no similar assessed questions.*

Additional Exercise E61 Challenging

The homomorphism

$$\begin{aligned}\phi : (\mathbb{C}^*, \times) &\longrightarrow (\mathbb{C}^*, \times)\\ z &\longmapsto z/|z|\end{aligned}$$

has image

$$\operatorname{Im}\phi = \{z \in \mathbb{C} : |z| = 1\}$$

and kernel

$$\operatorname{Ker}\phi = \mathbb{R}^+.$$

(This mapping was shown to be a homomorphism and its image and kernel were found in the solution to Additional Exercise E57. In the solution to Additional Exercise E60 a group of real numbers isomorphic to $\mathbb{C}^*/\operatorname{Ker}\phi$ was found.)

(a) Show that the coset $(1+i)\operatorname{Ker}\phi$ is the set of all complex numbers with principal argument $\pi/4$, and describe this set geometrically as a subset of the complex plane.

(b) Express the general coset $(a+ib)\operatorname{Ker}\phi$ where $a+ib \in \mathbb{C}^*$ in set notation, and describe it geometrically as a subset of the complex plane.

(c) Hence specify the elements of the quotient group $\mathbb{C}^*/\operatorname{Ker}\phi$ and describe them geometrically.

Additional Exercise E62 Challenging

The homomorphism

$$\begin{aligned}\phi : (\mathbb{C}^*, \times) &\longrightarrow (\mathbb{C}^*, \times)\\ z &\longmapsto z^4\end{aligned}$$

has image

$$\operatorname{Im}\phi = \mathbb{C}^*$$

and kernel

$$\operatorname{Ker}\phi = \{1, i, -1, -i\}.$$

(This mapping was shown to be a homomorphism and its image and kernel and a standard group isomorphic to $\mathbb{C}^*/\operatorname{Ker}\phi$ were all found in the solution to Additional Exercise E58.)

(a) Determine the coset $(2+3i)\operatorname{Ker}\phi$, and describe it geometrically as a subset of the complex plane.

(b) Determine the general coset $(a+ib)\operatorname{Ker}\phi$ where $a+ib \in \mathbb{C}^*$, and describe it geometrically as a subset of the complex plane.

(c) Hence specify the elements of the quotient group $\mathbb{C}^*/\operatorname{Ker}\phi$ and describe them geometrically.

Additional Exercise E63 Challenging

The homomorphism

$$\begin{aligned}\phi : (L, \times) &\longrightarrow (D, \times)\\ \begin{pmatrix} a & 0\\ c & d\end{pmatrix} &\longmapsto \begin{pmatrix} a & 0\\ 0 & d\end{pmatrix},\end{aligned}$$

where L is the set of all invertible 2×2 lower triangular matrices, has kernel

$$\operatorname{Ker}\phi = \left\{\begin{pmatrix} 1 & 0\\ c & 1\end{pmatrix} : c \in \mathbb{R}\right\}.$$

(This mapping was shown to be a homomorphism in the solution to Exercise E109(a) in Subsection 1.2 of Unit E3. Its kernel was found in the solution to Exercise E124 in Subsection 2.3 of Unit E3.)

(a) Determine the particular coset

$$\begin{pmatrix} 1 & 0\\ 2 & 3\end{pmatrix}\operatorname{Ker}\phi.$$

(b) Determine the general coset

$$\begin{pmatrix} x & 0\\ y & z\end{pmatrix}\operatorname{Ker}\phi.$$

(c) Show that (as we should expect from Theorem E54 in Unit E3) the image under ϕ of any element of the coset in part (b) is the same as the image under ϕ of the particular element

$$\begin{pmatrix} x & 0\\ y & z\end{pmatrix}$$

of this coset.

Solutions to additional exercises for Unit E3

Solution to Additional Exercise E41

We first find a generator of each group.

To find a generator of $(\mathbb{Z}_7^*, \times_7)$, we try possibilities until we find one. First we determine whether 2 is a generator. The consecutive powers of 2 in $(\mathbb{Z}_7^*, \times_7)$ starting from 2^0 are

$$1, 2, 4, 1, \ldots.$$

This list does not contain every element of $(\mathbb{Z}_7^*, \times_7)$, so 2 is not a generator of $(\mathbb{Z}_7^*, \times_7)$.

Now we determine whether 3 is a generator. The consecutive powers of 3 in $(\mathbb{Z}_7^*, \times_7)$ starting from 3^0 are

$$1, 3, 2, 6, 4, 5 \ldots.$$

This list contains every element of $(\mathbb{Z}_7^*, \times_7)$, so 3 is a generator of $(\mathbb{Z}_7^*, \times_7)$.

We know that 1 is a generator of $(\mathbb{Z}_6, +_6)$. The consecutive multiples of 1 in $(\mathbb{Z}_6, +_6)$ starting from $0(1)$ are

$$0, 1, 2, 3, 4, 5, \ldots.$$

Matching each power of the generator 3 of $(\mathbb{Z}_7^*, \times_7)$ with the corresponding multiple of the generator 1 of $(\mathbb{Z}_6, +_6)$ gives the following isomorphism:

$$\begin{aligned}\phi : (\mathbb{Z}_7^*, \times_7) &\longrightarrow (\mathbb{Z}_6, +_6)\\ 1 &\longmapsto 0\\ 3 &\longmapsto 1\\ 2 &\longmapsto 2\\ 6 &\longmapsto 3\\ 4 &\longmapsto 4\\ 5 &\longmapsto 5.\end{aligned}$$

(Another generator of $(\mathbb{Z}_6, +_6)$ is 5, the inverse of 1. Matching each power of the generator 3 of $(\mathbb{Z}_7^*, \times_7)$ with the corresponding multiple of the generator 5 of $(\mathbb{Z}_6, +_6)$ gives the following isomorphism:

$$\begin{aligned}\phi : (\mathbb{Z}_7^*, \times_7) &\longrightarrow (\mathbb{Z}_6, +_6)\\ 1 &\longmapsto 0\\ 3 &\longmapsto 5\\ 2 &\longmapsto 4\\ 6 &\longmapsto 3\\ 4 &\longmapsto 2\\ 5 &\longmapsto 1.\end{aligned}$$

There are no other generators of $(\mathbb{Z}_6, +_6)$ and hence no other isomorphisms from $(\mathbb{Z}_7^*, \times_7)$ to $(\mathbb{Z}_6, +_6)$.)

Solution to Additional Exercise E42

We have

$$U_9 = \{1, 2, 4, 5, 7, 8\}$$

and

$$U_{14} = \{1, 3, 5, 9, 11, 13\}.$$

So both $(U_9, \times_9)$ and $(U_{14}, \times_{14})$ have order 6.

To show that these groups are cyclic, we have to find a generator of each of them.

In $(U_9, \times_9)$ the consecutive powers of 2 starting from 2^0 are

$$1, 2, 4, 8, 7, 5, \ldots.$$

Since all the elements of U_9 appear in this list, $(U_9, \times_9)$ is cyclic, generated by 2.

In $(U_{14}, \times_{14})$ the consecutive powers of 3 starting from 3^0 are

$$1, 3, 9, 13, 11, 5, \ldots.$$

Since all the elements of U_{14} appear in this list, $(U_{14}, \times_{14})$ is cyclic, generated by 3.

Matching the powers of the generators in each group gives the following isomorphism:

$$\begin{aligned}\phi : (U_9, \times_9) &\longrightarrow (U_{14}, \times_{14})\\ 1 &\longmapsto 1\\ 2 &\longmapsto 3\\ 4 &\longmapsto 9\\ 8 &\longmapsto 13\\ 7 &\longmapsto 11\\ 5 &\longmapsto 5.\end{aligned}$$

(The group $(U_{14}, \times_{14})$ is also generated by $3^{-1} = 5$, so an alternative answer is

$$\begin{aligned}\phi : (U_9, \times_9) &\longrightarrow (U_{14}, \times_{14})\\ 1 &\longmapsto 1\\ 2 &\longmapsto 5\\ 4 &\longmapsto 11\\ 8 &\longmapsto 13\\ 7 &\longmapsto 9\\ 5 &\longmapsto 3.\end{aligned}$$

There are no other isomorphisms from $(U_9, \times_9)$ to $(U_{14}, \times_{14})$.)

Solution to Additional Exercise E43

Let $f, g \in S(\triangle)$. We have to show that

$$\phi(f \circ g) = \phi(f) \circ \phi(g).$$

We know that a composite of two direct symmetries or two indirect symmetries is direct, and a composite of a direct symmetry and an indirect symmetry is indirect.

If f is direct and g is direct then $f \circ g$ is direct so

$$\phi(f \circ g) = e \quad \text{and} \quad \phi(f) \circ \phi(g) = e \circ e = e.$$

If f is direct and g is indirect then $f \circ g$ is indirect so

$$\phi(f \circ g) = r \quad \text{and} \quad \phi(f) \circ \phi(g) = e \circ r = r.$$

If f is indirect and g is direct then $f \circ g$ is indirect so

$$\phi(f \circ g) = r \quad \text{and} \quad \phi(f) \circ \phi(g) = r \circ e = r.$$

If f is indirect and g is indirect then $f \circ g$ is direct so

$$\phi(f \circ g) = e \quad \text{and} \quad \phi(f) \circ \phi(g) = r \circ r = e.$$

Thus in all cases $\phi(f \circ g) = \phi(f) \circ \phi(g)$. Hence ϕ is a homomorphism.

Solution to Additional Exercise E44

(a) Let $x, y \in \mathbb{R}^*$. We have to show that

$$\phi(x \times y) = \phi(x) \times \phi(y).$$

Now

$$\begin{aligned}\phi(x \times y) &= (x \times y)^4 \\ &= x^4 \times y^4 \\ &= \phi(x) \times \phi(y).\end{aligned}$$

Hence ϕ is a homomorphism.

(b) Let $x, y \in \mathbb{R}^*$. We have to show that

$$\phi(x \times y) = \phi(x) \times \phi(y).$$

Now

$$\begin{aligned}\phi(x \times y) &= (x \times y)^5 \\ &= x^5 \times y^5 \\ &= \phi(x) \times \phi(y).\end{aligned}$$

Hence ϕ is a homomorphism.

(c) Let $w, z \in \mathbb{C}^*$. We have to show that

$$\phi(w \times z) = \phi(w) \times \phi(z).$$

Now

$$\begin{aligned}\phi(w \times z) &= \overline{w \times z} \\ &= \overline{w} \times \overline{z} \\ &= \phi(w) \times \phi(z).\end{aligned}$$

Hence ϕ is a homomorphism.

(d) Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. We have to show that

$$\phi((x_1, y_1) + (x_2, y_2)) = \phi(x_1, y_1) + \phi(x_2, y_2).$$

Now

$$\begin{aligned}\phi((x_1, y_1) + (x_2, y_2)) &= \phi(x_1 + x_2, y_1 + y_2) \\ &= (x_1 + x_2) - 2(y_1 + y_2) \\ &= (x_1 - 2y_1) + (x_2 - 2y_2) \\ &= \phi(x_1, y_1) + \phi(x_2, y_2).\end{aligned}$$

Hence ϕ is a homomorphism.

(In fact, ϕ is a linear transformation so we know that it is a homomorphism by Proposition E39 in Unit E3.)

(Recall that for simplicity we write $\phi(x, y)$ for $\phi((x, y))$, as in Unit C3.)

Solution to Additional Exercise E45

(a) The mapping ϕ is not one-to-one. For example, the elements -1 and 1 of the domain group are both mapped by ϕ to the element 1 of the codomain group. Hence ϕ is not an isomorphism.

(The mapping ϕ is also not onto. For example, the element -1 of the codomain group is not the image under ϕ of any element of the domain group.)

(b) We show that ϕ is one-to-one. Let $x, y \in \mathbb{R}^*$ and suppose that

$$\phi(x) = \phi(y),$$

that is,

$$x^5 = y^5.$$

It follows that $x = y$. Hence ϕ is one-to-one.

Also, ϕ is onto, since each positive element x of the codomain group is the image under ϕ of the element $x^{1/5}$ of the domain group, and each negative element x of the codomain group is the image under ϕ of the element $-(-x)^{1/5}$ of the domain group.

Finally, ϕ has the homomorphism property, by the solution to Additional Exercise E44(b).

Hence ϕ is an isomorphism.

(c) We show that ϕ is one-to-one. Let $w, z \in \mathbb{C}^*$ and suppose that

$$\phi(w) = \phi(z),$$

that is,

$$\overline{w} = \overline{z}.$$

It follows that $w = z$. Hence ϕ is one-to-one.

Also, ϕ is onto, since each element z of the codomain group is the image under ϕ of the element $\overline{z}$ of the domain group (since $\overline{(\overline{z})} = z$).

Finally, ϕ has the homomorphism property, by the solution to Additional Exercise E44(c).

Hence ϕ is an isomorphism.

(d) The mapping ϕ is not one-to-one. For example, the elements $(2, 1)$ and $(4, 2)$ of the domain group are both mapped by ϕ to the element 0 of the codomain group. Hence ϕ is not an isomorphism.

Solution to Additional Exercise E46

Let $\mathbf{A}, \mathbf{B} \in L$. We have to show that

$$\phi(\mathbf{AB}) = \phi(\mathbf{A})\phi(\mathbf{B}).$$

Now

$$\mathbf{A} = \begin{pmatrix} r & 0 \\ t & u \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} v & 0 \\ x & y \end{pmatrix},$$

for some $r, t, u, v, x, y \in \mathbb{R}$ with $ru \neq 0$ and $vy \neq 0$.

Hence

$$\begin{aligned} \phi(\mathbf{AB}) &= \phi\left(\begin{pmatrix} r & 0 \\ t & u \end{pmatrix}\begin{pmatrix} v & 0 \\ x & y \end{pmatrix}\right) \\ &= \phi\begin{pmatrix} rv & 0 \\ tv + ux & uy \end{pmatrix} \\ &= \begin{pmatrix} rv & 0 \\ 0 & rv \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \phi(\mathbf{A})\phi(\mathbf{B}) &= \phi\begin{pmatrix} r & 0 \\ t & u \end{pmatrix}\phi\begin{pmatrix} v & 0 \\ x & y \end{pmatrix} \\ &= \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}\begin{pmatrix} v & 0 \\ 0 & v \end{pmatrix} \\ &= \begin{pmatrix} rv & 0 \\ 0 & rv \end{pmatrix}. \end{aligned}$$

Thus $\phi(\mathbf{AB}) = \phi(\mathbf{A})\phi(\mathbf{B})$. Hence ϕ is a homomorphism.

Solution to Additional Exercise E47

Let $\mathbf{A}, \mathbf{B} \in M_{2,2}$. We have to show that

$$\phi(\mathbf{A} + \mathbf{B}) = \phi(\mathbf{A}) + \phi(\mathbf{B}).$$

Now

$$\mathbf{A} = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} v & w \\ x & y \end{pmatrix},$$

for some $r, s, t, u, v, w, x, y \in \mathbb{R}$.

Hence

$$\begin{aligned} \phi(\mathbf{A} + \mathbf{B}) &= \phi\left(\begin{pmatrix} r & s \\ t & u \end{pmatrix} + \begin{pmatrix} v & w \\ x & y \end{pmatrix}\right) \\ &= \phi\begin{pmatrix} r + v & s + w \\ t + x & u + y \end{pmatrix} \\ &= r + v + u + y \end{aligned}$$

and

$$\begin{aligned} \phi(\mathbf{A}) + \phi(\mathbf{B}) &= \phi\begin{pmatrix} r & s \\ t & u \end{pmatrix} + \phi\begin{pmatrix} v & w \\ x & y \end{pmatrix} \\ &= (r + u) + (v + y) \\ &= r + v + u + y. \end{aligned}$$

Thus $\phi(\mathbf{A} + \mathbf{B}) = \phi(\mathbf{A}) + \phi(\mathbf{B})$. Hence ϕ is a homomorphism.

Solution to Additional Exercise E48

(a) The homomorphism property for ϕ is

$$\phi((x_1, y_1) + (x_2, y_2)) = \phi(x_1, y_1) + \phi(x_2, y_2)$$
for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$.

The mapping ϕ does not have this property. For example, $(1, 0), (0, 1) \in \mathbb{R}^2$ and

$$\phi((1, 0) + (0, 1)) = \phi(1, 1) = 3,$$

whereas

$$\phi(1, 0) + \phi(0, 1) = 2 + 2 = 4.$$

Thus

$$\phi((1, 0) + (0, 1)) \neq \phi(1, 0) + \phi(0, 1).$$

Hence ϕ is not a homomorphism.

(An alternative way to show that this mapping ϕ is not a homomorphism is to note that the identity elements of the domain group and codomain group are $(0, 0)$ and 0, respectively, but

$$\phi(0, 0) = 0 + 0 + 1 = 1 \neq 0.$$

Hence ϕ is not a homomorphism by Proposition E41 in Unit E3.)

(b) The homomorphism property for ϕ is

$$\phi(x+y)=\phi(x)+\phi(y) \quad \text{for all } x,y\in\mathbb{R}.$$

The mapping ϕ does not have this property. For example, $1\in\mathbb{R}$ and

$$\phi(1+1)=\phi(2)=4,$$

but

$$\phi(1)+\phi(1)=1+1=2.$$

Thus

$$\phi(1+1)\neq\phi(1)+\phi(1).$$

Hence ϕ is not a homomorphism.

Solution to Additional Exercise E49

We have to show that ϕ is one-to-one, onto and preserves composites.

To show that ϕ is one-to-one, let $x,y\in G$ and suppose that

$$\phi(x)=\phi(y).$$

Then

$$x^{-1}=y^{-1}.$$

Composing each side of this equation by x on the left and y on the right gives

$$x\circ x^{-1}\circ y=x\circ y^{-1}\circ y,$$

that is,

$$y=x.$$

Hence ϕ is one-to-one.

Also, ϕ is onto because, for each $x\in G$,

$$\begin{aligned}\phi(x^{-1})&=(x^{-1})^{-1}\\&=x.\end{aligned}$$

Finally we show that ϕ preserves composites. Let $x,y\in G$. We have to show that

$$\phi(x\circ y)=\phi(x)\circ\phi(y).$$

Now

$$\begin{aligned}\phi(x\circ y)&=(x\circ y)^{-1}\\&=y^{-1}\circ x^{-1}\\&=x^{-1}\circ y^{-1}\quad\text{(since } (G,\circ) \text{ is abelian)}\\&=\phi(x)\circ\phi(y).\end{aligned}$$

Hence ϕ is an isomorphism.

Solution to Additional Exercise E50

We have to show that ϕ_g is one-to-one, onto and preserves composites.

To show that ϕ_g is one-to-one, let $x,y\in G$ and suppose that

$$\phi_g(x)=\phi_g(y).$$

Then

$$g\circ x\circ g^{-1}=g\circ y\circ g^{-1}.$$

It follows by the Cancellation Laws that $x=y$. Hence ϕ_g is one-to-one.

Also, ϕ_g is onto because, for each $x\in G$,

$$\begin{aligned}\phi_g(g^{-1}\circ x\circ g)&=g\circ g^{-1}\circ x\circ g\circ g^{-1}\\&=e\circ x\circ e\\&=x.\end{aligned}$$

Finally we show that ϕ_g preserves composites. Let $x,y\in G$. We have to show that

$$\phi_g(x\circ y)=\phi_g(x)\circ\phi_g(y).$$

Now

$$\begin{aligned}\phi_g(x)\circ\phi_g(y)&=(g\circ x\circ g^{-1})\circ(g\circ y\circ g^{-1})\\&=g\circ x\circ(g^{-1}\circ g)\circ y\circ g^{-1}\\&=g\circ x\circ y\circ g^{-1}\\&=\phi_g(x\circ y).\end{aligned}$$

Hence ϕ_g is an isomorphism.

Solution to Additional Exercise E51

We check the group axioms for $(A,\circ)$.

G1 Let $f,g\in A$. We have to show that $f\circ g\in A$, that is, we have to show that $f\circ g$ is an automorphism of $(G,*)$.

Since both f and g are one-to-one and onto mappings from G to G, so is $f\circ g$.

To show that $f\circ g$ has the homomorphism property, let $x,y\in G$. We have to show that

$$(f\circ g)(x*y)=(f\circ g)(x)*(f\circ g)(y).$$

Now

$$\begin{aligned}(f\circ g)(x*y)&=f(g(x*y))\\&=f(g(x)*g(y))\\&\qquad\text{(since } g \text{ is a homomorphism)}\\&=f(g(x))*f(g(y))\\&\qquad\text{(since } f \text{ is a homomorphism)}\\&=(f\circ g)(x)*(f\circ g)(y).\end{aligned}$$

So $f \circ g$ has the homomorphism property and hence it is an automorphism of $(G, *)$. Thus $(A, \circ)$ is closed under function composition.

G2 Function composition is an associative binary operation.

G3 The identity mapping, say i, from G to G is clearly an automorphism of $(G, *)$, so it is an element of A, and it has the property that

$$f \circ i = f = i \circ f,$$

for all $f \in A$. Thus i is an identity element for function composition on A.

G4 Let f be any element of A, that is, any automorphism of $(G, *)$. Since f is one-to-one its inverse f^{-1} exists and satisfies

$$f \circ f^{-1} = i = f^{-1} \circ f.$$

Also by Proposition E36 in Unit E3 the mapping f^{-1} is an automorphism of $(G, *)$, that is, it is an element of A. So f^{-1} is an inverse of f in A with respect to function composition.

This shows that every element of A has an inverse in A with respect to function composition.

Therefore $(A, \circ)$ satisfies the four group axioms, and so it is a group.

Solution to Additional Exercise E52

(a) We show that $\operatorname{Im}\phi = \mathbb{R}^+$.

We have that $x^4 > 0$ for all $x \in \mathbb{R}^*$, so $\operatorname{Im}\phi \subseteq \mathbb{R}^+$.

Also, each number r in $\mathbb{R}^+$ is the image under ϕ of $\sqrt[4]{r}$ (the positive fourth root of r), so $\mathbb{R}^+ \subseteq \operatorname{Im}\phi$.

Thus $\operatorname{Im}\phi = \mathbb{R}^+$, as claimed.

The identity element of $(\mathbb{R}^*, \times)$ is 1, so

$$\begin{aligned}\operatorname{Ker}\phi &= \{x \in \mathbb{R}^* : \phi(x) = 1\}\\ &= \{x \in \mathbb{R}^* : x^4 = 1\}\\ &= \{1, -1\}.\end{aligned}$$

(b) This mapping ϕ is an isomorphism, as shown in the solution to Additional Exercise E45(b).

Since ϕ is onto, its image is its whole codomain group, that is,

$$\operatorname{Im}\phi = \mathbb{R}^*.$$

Since ϕ is one-to-one, its kernel contains the identity element of the domain group alone, by Theorem E52 in Unit E3. So

$$\operatorname{Ker}\phi = \{1\}.$$

(c) This mapping ϕ is an isomorphism, as shown in the solution to Additional Exercise E45(c). Hence, by arguments similar to those in part (b),

$$\operatorname{Im}\phi = \mathbb{C}^*$$

and

$$\operatorname{Ker}\phi = \{1\}.$$

(d) Each real number r is the image under ϕ of the element $(r, 0)$ of $\mathbb{R}^2$. So ϕ is onto and hence

$$\operatorname{Im}\phi = \mathbb{R}.$$

The identity element of $(\mathbb{R}, +)$ is 0, so

$$\begin{aligned}\operatorname{Ker}\phi &= \{(x, y) \in \mathbb{R}^2 : \phi(x, y) = 0\}\\ &= \{(x, y) \in \mathbb{R}^2 : x - 2y = 0\}\\ &= \{(x, y) \in \mathbb{R}^2 : y = \tfrac{1}{2}x\}\\ &= \{(2k, k) : k \in \mathbb{R}\}.\end{aligned}$$

(Any of the last three lines above is a suitable description of the kernel. It can also be described as the line $y = \frac{1}{2}x$.)

Solution to Additional Exercise E53

For this mapping,

$$\begin{aligned}\operatorname{Im}\phi &= \left\{\phi\begin{pmatrix} a & 0\\ c & d\end{pmatrix} : \begin{pmatrix} a & 0\\ c & d\end{pmatrix} \in L\right\}\\ &= \left\{\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} : a \in \mathbb{R}^*\right\}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Ker}\phi &= \left\{\begin{pmatrix} a & 0\\ c & d\end{pmatrix} \in L : \phi\begin{pmatrix} a & 0\\ c & d\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right\}\\ &= \left\{\begin{pmatrix} a & 0\\ c & d\end{pmatrix} \in L : \begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} = \begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\right\}\\ &= \left\{\begin{pmatrix} a & 0\\ c & d\end{pmatrix} \in L : a = 1\right\}\\ &= \left\{\begin{pmatrix} 1 & 0\\ c & d\end{pmatrix} : c, d \in \mathbb{R},\ d \neq 0\right\}.\end{aligned}$$

Solution to Additional Exercise E54

Let $\mathbf{A}, \mathbf{B} \in L$. We have to show that

$$\phi(\mathbf{AB}) = \phi(\mathbf{A})\phi(\mathbf{B}).$$

Now

$$\mathbf{A} = \begin{pmatrix} r & 0 \\ t & u \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} v & 0 \\ x & y \end{pmatrix},$$

for some $r, t, u, v, x, y \in \mathbb{R}$ with $ru \neq 0$ and $vy \neq 0$. Hence

$$\begin{aligned} \phi(\mathbf{AB}) &= \phi\left(\begin{pmatrix} r & 0 \\ t & u \end{pmatrix}\begin{pmatrix} v & 0 \\ x & y \end{pmatrix}\right) \\ &= \phi\begin{pmatrix} rv & 0 \\ tv + ux & uy \end{pmatrix} \\ &= rv \end{aligned}$$

and

$$\begin{aligned} \phi(\mathbf{A})\phi(\mathbf{B}) &= \phi\begin{pmatrix} r & 0 \\ t & u \end{pmatrix}\phi\begin{pmatrix} v & 0 \\ x & y \end{pmatrix} \\ &= rv. \end{aligned}$$

Thus $\phi(\mathbf{AB}) = \phi(\mathbf{A})\phi(\mathbf{B})$. Hence ϕ is a homomorphism.

The mapping ϕ is onto because if $r \in \mathbb{R}^*$, then

$$\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \in L \quad \text{and} \quad \phi\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} = r.$$

Hence

$$\operatorname{Im}\phi = \mathbb{R}^*.$$

The identity element of the codomain group is 1. So

$$\begin{aligned} \operatorname{Ker}\phi &= \left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in L : \phi\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} = 1\right\} \\ &= \left\{\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in L : a = 1\right\} \\ &= \left\{\begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} : c, d \in \mathbb{R},\ d \neq 0\right\}. \end{aligned}$$

Solution to Additional Exercise E55

The mapping ϕ is onto because if $r \in \mathbb{R}$ then

$$\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} \in M_{2,2} \quad \text{and} \quad \phi\begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix} = r.$$

Hence

$$\operatorname{Im}\phi = \mathbb{R}.$$

The identity element in $(\mathbb{R}, +)$ is 0, so

$$\begin{aligned} \operatorname{Ker}\phi &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2} : \phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 0\right\} \\ &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_{2,2} : a + d = 0\right\} \\ &= \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ d = -a\right\} \\ &= \left\{\begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{R}\right\}. \end{aligned}$$

Solution to Additional Exercise E56

(a) The set M is a *subset* of the group $\mathrm{GL}(2)$, because if

$$\mathbf{A} = \begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

is a matrix in M, then $a, b \in \mathbb{R}$ and $\mathbf{A}$ is invertible because $\det \mathbf{A} = a^2 - b^2 \neq 0$.

Also, the binary operation specified for M is the same as the binary operation of $\mathrm{GL}(2)$.

We now show that the three subgroup properties hold for M.

SG1 Let $\mathbf{A}, \mathbf{B} \in M$. Then

$$\mathbf{A} = \begin{pmatrix} r & s \\ s & r \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} t & u \\ u & t \end{pmatrix},$$

where $r, s, t, u \in \mathbb{R}$ with $r^2 - s^2 \neq 0$ and $t^2 - u^2 \neq 0$. So

$$\mathbf{AB} = \begin{pmatrix} r & s \\ s & r \end{pmatrix}\begin{pmatrix} t & u \\ u & t \end{pmatrix} = \begin{pmatrix} rt + su & ru + st \\ st + ru & su + rt \end{pmatrix}.$$

This matrix is of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with $a = rt + su$ and $b = ru + st$. It also satisfies the condition $a^2 - b^2 \neq 0$ because this condition is equivalent to the condition $\det(\mathbf{AB}) \neq 0$, and every element of $\mathrm{GL}(2)$ has non-zero determinant. (We know that $\mathbf{AB} \in \mathrm{GL}(2)$ because $\mathbf{A}, \mathbf{B} \in \mathrm{GL}(2)$.) Hence $\mathbf{AB} \in M$.

Thus M is closed.

SG2 The identity matrix

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

belongs to M, since it is of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with $a = 1$ and $b = 0$, and
$a^2 - b^2 = 1^2 - 0^2 = 1 \neq 0$.

SG3 Let $\mathbf{A} \in M$. Then

$$\mathbf{A} = \begin{pmatrix} r & s \\ s & r \end{pmatrix}$$

where $r, s \in \mathbb{R}$ and $r^2 - s^2 \neq 0$. The inverse of this matrix in GL(2) is

$$\begin{aligned} \mathbf{A}^{-1} &= \frac{1}{r^2 - s^2} \begin{pmatrix} r & -s \\ -s & r \end{pmatrix} \\ &= \begin{pmatrix} r/(r^2 - s^2) & -s/(r^2 - s^2) \\ -s/(r^2 - s^2) & r/(r^2 - s^2) \end{pmatrix}. \end{aligned}$$

This matrix is of the form

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix}$$

with $a = r/(r^2 - s^2)$ and $b = -s/(r^2 - s^2)$. It also satisfies the condition $a^2 - b^2 \neq 0$ because this condition is equivalent to the condition $\det(\mathbf{A}^{-1}) \neq 0$, and every element of GL(2) has non-zero determinant. (We know that $\mathbf{A}^{-1} \in \mathrm{GL}(2)$ because $\mathbf{A} \in \mathrm{GL}(2)$.) Hence $\mathbf{A}^{-1} \in M$.

Thus M contains the inverse of each of its elements.

Hence M satisfies the three subgroup properties and is therefore a subgroup of GL(2).

(b) Let $\mathbf{A}, \mathbf{B} \in M$. We have to show that

$$\phi(\mathbf{AB}) = \phi(\mathbf{A})\phi(\mathbf{B}).$$

Now

$$\mathbf{A} = \begin{pmatrix} r & s \\ s & r \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} t & u \\ u & t \end{pmatrix},$$

where $r, s, t, u \in \mathbb{R}$ with $r^2 - s^2 \neq 0$ and $t^2 - u^2 \neq 0$.

Hence

$$\begin{aligned} \phi(\mathbf{AB}) &= \phi\left(\begin{pmatrix} r & s \\ s & r \end{pmatrix}\begin{pmatrix} t & u \\ u & t \end{pmatrix}\right) \\ &= \phi\begin{pmatrix} rt + su & ru + st \\ st + ru & su + rt \end{pmatrix} \\ &= (rt + su) - (ru + st) \\ &= rt + su - ru - st \end{aligned}$$

and

$$\begin{aligned} \phi(\mathbf{A})\phi(\mathbf{B}) &= (r - s)(t - u) \\ &= rt - ru - st + su \\ &= rt + su - ru - st. \end{aligned}$$

Thus $\phi(\mathbf{AB}) = \phi(\mathbf{A})\phi(\mathbf{B})$. Hence ϕ is a homomorphism.

(c) If $r \in \mathbb{R}^*$ then

$$\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \in M \quad \text{and} \quad \phi\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = r - 0 = r.$$

Thus ϕ is onto.

The identity element of $(\mathbb{R}^*, \times)$ is 1, so

$$\begin{aligned} &\operatorname{Ker}\phi \\ &= \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} \in M : \phi\begin{pmatrix} a & b \\ b & a \end{pmatrix} = 1\right\} \\ &= \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a^2 - b^2 \neq 0,\ a - b = 1\right\} \\ &= \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a + b \neq 0,\ a - b = 1\right\} \\ &\qquad \text{(since } a^2 - b^2 = (a - b)(a + b) \text{ and } a - b \neq 0) \\ &= \left\{\begin{pmatrix} a & b \\ b & a \end{pmatrix} : a, b \in \mathbb{R},\ a + b \neq 0,\ b = a - 1\right\} \\ &= \left\{\begin{pmatrix} a & a - 1 \\ a - 1 & a \end{pmatrix} : a \in \mathbb{R},\ a \neq \tfrac{1}{2}\right\} \\ &\qquad \text{(since } a + (a - 1) \neq 0 \text{ gives } a \neq \tfrac{1}{2}). \end{aligned}$$

Solution to Additional Exercise E57

Let z, $w \in \mathbb{C}^*$. Then

$$\begin{aligned}\phi(zw) &= \frac{zw}{|zw|}\\ &= \frac{zw}{|z||w|}\\ &= \frac{z}{|z|} \times \frac{w}{|w|}\\ &= \phi(z)\phi(w).\end{aligned}$$

Thus ϕ is a homomorphism.

From the definition of ϕ, it seems likely that $\operatorname{Im}\phi$ consists of the complex numbers with modulus 1, that is, $\operatorname{Im}\phi = \{z \in \mathbb{C} : |z| = 1\}$. We now prove this.

For any $z \in \mathbb{C}^*$,

$$|\phi(z)| = \left|\frac{z}{|z|}\right| = \frac{|z|}{|z|} = 1.$$

Hence

$$\operatorname{Im}\phi \subseteq \{z \in \mathbb{C} : |z| = 1\}.$$

Also, if $|z| = 1$, then $z \in \mathbb{C}^*$ and $\phi(z) = z$, so $z \in \operatorname{Im}\phi$. Hence

$$\{z \in \mathbb{C} : |z| = 1\} \subseteq \operatorname{Im}\phi.$$

Therefore

$$\operatorname{Im}\phi = \{z \in \mathbb{C} : |z| = 1\}.$$

(This is the circle with centre 0 and radius 1 in the complex plane.)

The identity element in $(\mathbb{C}^*, \times)$ is 1, so

$$\begin{aligned}\operatorname{Ker}\phi &= \{z \in \mathbb{C}^* : \phi(z) = 1\}\\ &= \left\{z \in \mathbb{C}^* : \frac{z}{|z|} = 1\right\}\\ &= \{z \in \mathbb{C}^* : z = |z|\}\\ &= \mathbb{R}^+.\end{aligned}$$

Solution to Additional Exercise E58

(a) For all $z, w \in \mathbb{C}^*$,

$$\phi(z \times w) = (z \times w)^4 = z^4 \times w^4 = \phi(z) \times \phi(w).$$

Hence ϕ is a homomorphism.

(b) The homomorphism ϕ is onto, because each element $w \in \mathbb{C}^*$ is the image under ϕ of any of the four fourth roots of w in $\mathbb{C}^*$. Thus

$$\operatorname{Im}\phi = \mathbb{C}^*.$$

The identity element in $(\mathbb{C}^*, \times)$ is 1, so

$$\begin{aligned}\operatorname{Ker}\phi &= \{z \in \mathbb{C}^* : \phi(z) = 1\}\\ &= \{z \in \mathbb{C}^* : z^4 = 1\}\\ &= \{1, i, -1, -i\}.\end{aligned}$$

(c) By the First Isomorphism Theorem,

$$\mathbb{C}^*/\operatorname{Ker}\phi \cong \operatorname{Im}\phi = (\mathbb{C}^*, \times).$$

So a standard group isomorphic to $\mathbb{C}^*/\operatorname{Ker}\phi$ is $(\mathbb{C}^*, \times)$.

Solution to Additional Exercise E59

By the First Isomorphism Theorem, the quotient group $L/\operatorname{Ker}\phi$ is isomorphic to

$$\operatorname{Im}\phi = \left\{\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} : a \in \mathbb{R}^*\right\}.$$

From this specification of $\operatorname{Im}\phi$, we expect that $\operatorname{Im}\phi \cong (\mathbb{R}^*, \times)$. We now confirm this.

Consider the mapping

$$\begin{aligned}\theta : \operatorname{Im}\phi &\longrightarrow \mathbb{R}^*\\ \begin{pmatrix} a & 0\\ 0 & a\end{pmatrix} &\longmapsto a.\end{aligned}$$

This mapping θ is one-to-one, because if

$$\mathbf{A} = \begin{pmatrix} r & 0\\ 0 & r\end{pmatrix} \quad\text{and}\quad \mathbf{B} = \begin{pmatrix} s & 0\\ 0 & s\end{pmatrix}$$

are elements of $\operatorname{Im}\phi$ such that

$$\theta(\mathbf{A}) = \theta(\mathbf{B}),$$

then $r = s$.

The mapping θ is also onto, because if $r \in \mathbb{R}^*$ then

$$\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix} \in \operatorname{Im}\phi \quad\text{and}\quad \theta\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix} = r.$$

Finally, θ has the homomorphism property, since for all $r, s \in \mathbb{R}^*$,

$$\begin{aligned}\theta\left(\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix}\begin{pmatrix} s & 0\\ 0 & s\end{pmatrix}\right) &= \theta\begin{pmatrix} rs & 0\\ 0 & rs\end{pmatrix}\\ &= rs\\ &= \theta\begin{pmatrix} r & 0\\ 0 & r\end{pmatrix}\theta\begin{pmatrix} s & 0\\ 0 & s\end{pmatrix}.\end{aligned}$$

Thus θ is an isomorphism.

It follows that

$$L/\operatorname{Ker}\phi \cong \operatorname{Im}\phi \cong (\mathbb{R}^*, \times).$$

So a standard group isomorphic to $L/\operatorname{Ker}\phi$ is $(\mathbb{R}^*, \times)$.

Solution to Additional Exercise E60

By the First Isomorphism Theorem, the quotient group $\mathbb{C}^*/\operatorname{Ker}\phi$ is isomorphic to

$$\operatorname{Im}\phi = \{z \in \mathbb{C} : |z| = 1\}.$$

We now show that $\operatorname{Im}\phi \cong ((-\pi, \pi], +_{2\pi})$.

Consider the mapping

$$\begin{aligned} \theta : \operatorname{Im}\phi &\longrightarrow ((-\pi, \pi], +_{2\pi}) \\ z &\longmapsto \operatorname{Arg} z. \end{aligned}$$

To show that this mapping is one-to-one, let $z, w \in \operatorname{Im}\phi$ and suppose that

$$\theta(z) = \theta(w).$$

This gives

$$\operatorname{Arg} z = \operatorname{Arg} w.$$

So z and w have the same principal argument. They also both have modulus 1, since they are both in $\operatorname{Im}\phi$. Hence $z = w$. Therefore θ is one-to-one.

Also, θ is onto, because if $r \in (-\pi, \pi]$ then the complex number z with modulus 1 and principal argument r lies in $\operatorname{Im}\phi$ and $\theta(z) = r$.

Finally, θ has the homomorphism property, since for all $z, w \in \operatorname{Im}\phi$,

$$\begin{aligned} \theta(z \times w) &= \operatorname{Arg}(z \times w) \\ &= \operatorname{Arg} z +_{2\pi} \operatorname{Arg} w \\ &= \theta(z) +_{2\pi} \theta(w). \end{aligned}$$

Thus

$$\mathbb{C}^*/\operatorname{Ker}\phi \cong \operatorname{Im}\phi \cong ((-\pi, \pi], +_{2\pi}).$$

Hence $\mathbb{C}^*/\operatorname{Ker}\phi$ is isomorphic to $((-\pi, \pi], +_{2\pi})$.

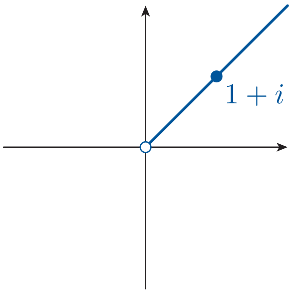
Solution to Additional Exercise E61

(a) We have

$$(1+i)\operatorname{Ker}\phi = \{(1+i)r : r \in \mathbb{R}^+\}.$$

This is the set of all complex numbers with the same principal argument as $1+i$, namely $\pi/4$.

In the complex plane, this set consists of 'half' of the line that passes through the number $1+i$, namely the 'half' that lies on the same side of the origin as $1+i$, as shown below.



(b) Similarly,

$$\begin{aligned} (a+ib)\operatorname{Ker}\phi &= \{(a+ib)r : r \in \mathbb{R}^+\} \\ &= \{z \in \mathbb{C}^* : \operatorname{Arg} z = \operatorname{Arg}(a+ib)\}. \end{aligned}$$

This is the set of all complex numbers with the same principal argument as $a+ib$.

In the complex plane, this set consists of 'half' of the line that passes through the number $a+ib$, namely the 'half' that lies on the same side of the origin as $a+ib$.

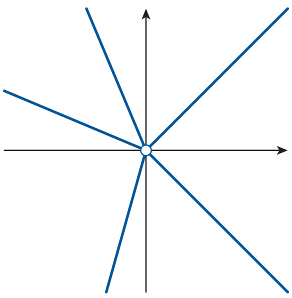
(Such a set is called a *half-line* or a *ray*.)

(c) The elements of the quotient group $\mathbb{C}^*/\operatorname{Ker}\phi$ are the cosets of $\operatorname{Ker}\phi$ in $(\mathbb{C}, *)$. By part (b), the cosets of $\operatorname{Ker}\phi$ are the sets of the form

$$\{z \in \mathbb{C}^* : \operatorname{Arg} z = \alpha\},$$

where $\alpha \in (-\pi, \pi]$.

Thus the cosets are the 'half-lines' that start at (but do not include) the origin, five of which are shown below.

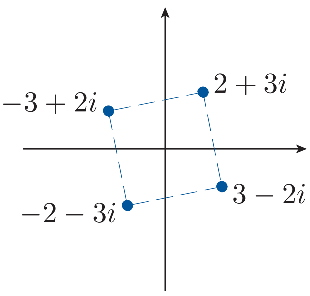


Solution to Additional Exercise E62

(a) We have

$$\begin{aligned}&(2+3i)\operatorname{Ker}\phi\\ &=\{(2+3i)\times 1,(2+3i)i,(2+3i)(-1),(2+3i)(-i)\}\\ &=\{2+3i,\ -3+2i,\ -2-3i,\ 3-2i\}.\end{aligned}$$

In the complex plane, these four complex numbers are the vertices of the square with centre at the origin and a vertex at $2+3i$. (There is only one such square, shown below.)

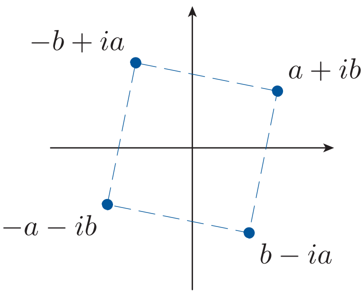


(b) We have

$$\begin{aligned}&(a+ib)\operatorname{Ker}\phi\\ &=\{(a+ib)\times 1,(a+ib)i,(a+ib)(-1),(a+ib)(-i)\}\\ &=\{a+ib,\ -b+ia,\ -a-ib,\ b-ia\}.\end{aligned}$$

In the complex plane, these four complex numbers are the vertices of the square with centre at the origin and a vertex at $a+ib$.

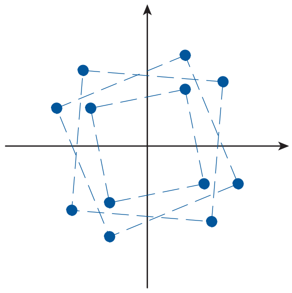
(An example is illustrated below.)



(c) The four vertices of every square with centre the origin form a coset of $\operatorname{Ker}\phi$, because they are the elements of the coset $(a+ib)\operatorname{Ker}\phi$, where $a+ib$ is any one vertex.

Thus the elements of $\mathbb{C}^*/\operatorname{Ker}\phi$ are the sets of vertices of squares with centre the origin.

(Three of these elements are shown together below.)



Solution to Additional Exercise E63

(a) This coset of $\operatorname{Ker}\phi$ is

$$\begin{aligned}\begin{pmatrix}1&0\\2&3\end{pmatrix}\operatorname{Ker}\phi&=\left\{\begin{pmatrix}1&0\\2&3\end{pmatrix}\begin{pmatrix}1&0\\c&1\end{pmatrix}:c\in\mathbb{R}\right\}\\ &=\left\{\begin{pmatrix}1&0\\2+3c&3\end{pmatrix}:c\in\mathbb{R}\right\}\\ &=\left\{\begin{pmatrix}1&0\\t&3\end{pmatrix}:t\in\mathbb{R}\right\}.\end{aligned}$$

(As c takes all values in $\mathbb{R}$, so does the expression $2+3c$, so we can replace it by t, where $t\in\mathbb{R}$.)

(b) This coset of $\operatorname{Ker}\phi$ is

$$\begin{aligned}\begin{pmatrix}x&0\\y&z\end{pmatrix}\operatorname{Ker}\phi&=\left\{\begin{pmatrix}x&0\\y&z\end{pmatrix}\begin{pmatrix}1&0\\c&1\end{pmatrix}:c\in\mathbb{R}\right\}\\ &=\left\{\begin{pmatrix}x&0\\y+zc&z\end{pmatrix}:c\in\mathbb{R}\right\}\\ &=\left\{\begin{pmatrix}x&0\\t&z\end{pmatrix}:t\in\mathbb{R}\right\}.\end{aligned}$$

(It follows from the definition of L that $z\neq 0$. Therefore, as c takes all values in $\mathbb{R}$, so does the expression $y+zc$, so we can replace it by t, where $t\in\mathbb{R}$.)

(c) For any $t\in\mathbb{R}$, we have

$$\phi\begin{pmatrix}x&0\\t&z\end{pmatrix}=\begin{pmatrix}x&0\\0&z\end{pmatrix}=\phi\begin{pmatrix}x&0\\y&z\end{pmatrix},$$

as expected.

Additional exercises for Unit E4

Section 1

Additional Exercise E64

Consider the symmetric group S_4 and the set

$$X = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}$$

of all unordered pairs (that is, subsets of size 2) of symbols from $\{1,2,3,4\}$.

The group S_4 has an obvious mapping effect $\wedge$ on the set X, given by

$$g \wedge \{i,j\} = \{g(i), g(j)\}$$

for all $g \in S_4$ and all $\{i,j\} \in X$. For example,

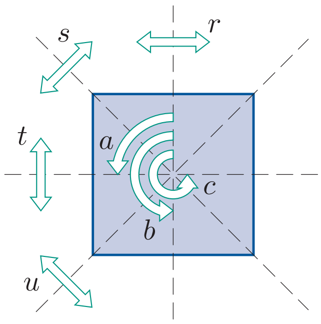
$$(1\ 3\ 4) \wedge \{1,3\} = \{3,4\}$$

since $(1\ 3\ 4)$ maps 1 to 3 and 3 to 4.

By checking the group action axioms, show that $\wedge$ is a group action of S_4 on X.

Additional Exercise E65

The non-identity symmetries of the square are shown below.

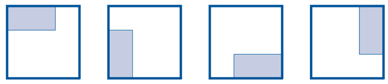


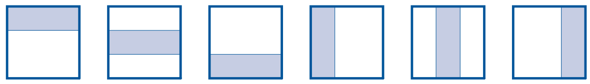
In each of parts (a) and (b) below, let X be the set of modified squares shown, and let $\wedge$ be the mapping effect of the group $S(\square)$ on the set X given by

$$g \wedge A = g(A)$$

for all $g \in S(\square)$ and all $A \in X$.

In each case, use Theorem E59 from Unit E4 to decide whether or not $\wedge$ is a group action. Where it is not a group action, show that it is not.

(a) 

(b) 

Additional Exercise E66

Consider the group $(\mathbb{R}^*, \times)$ and the set

$$X = \{(x,y) : x, y > 0\}.$$

(That is, X is the set of all points in the first quadrant of the plane.)

Let $\wedge$ be defined by

$$g \wedge (x,y) = (x^g, y)$$

for all $g \in \mathbb{R}^*$ and all $(x,y) \in X$.

Show that $\wedge$ is a group action.

Additional Exercise E67

Consider the group $(\mathbb{R}^+, \times)$ and the set

$$X = \{(x,y) : x, y > 0\}.$$

(That is, X is the set of all points in the first quadrant of the plane.)

In each of parts (a) and (b) below, let $\wedge$ be defined by the given equation for all $g \in \mathbb{R}^+$ and all $(x,y) \in X$. In each case show that $\wedge$ is not a group action.

(a) $g \wedge (x,y) = (g^x, y)$

(b) $g \wedge (x,y) = (gx^g, y)$

Additional Exercise E68

Let $\wedge$ be an action of a group $(G, \circ)$ on a set X, and let H be a subgroup of $(G, \circ)$. Let $\wedge_H$ be defined by

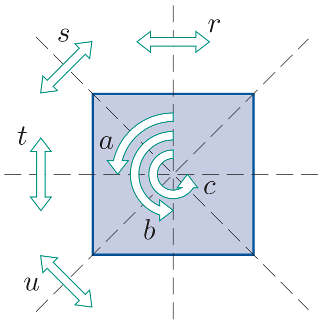
$$h \wedge_H x = h \wedge x$$

for all $h \in H$ and all $x \in X$. (In other words, $\wedge_H$ is the same as $\wedge$ except that the group elements involved are restricted to those of the subgroup H of G.) Show that $\wedge_H$ is an action of $(H, \circ)$ on X.

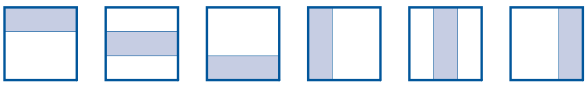
Section 2

Additional Exercise E69

The non-identity symmetries of the square are shown below.



Consider the action of the group $S(\square)$ on the set $\{A, B, C, D, E, F\}$ of modified squares shown below.



A B C D E F

(You saw that this is a group action in Additional Exercise E65.)

(a) Write down the orbits of this group action.

(b) Write down the stabiliser of each of the modified squares.

Additional Exercise E70

The solution to Exercise E142 in Subsection 1.3 of Unit E4 shows that the mapping effect $\wedge$ defined by

$$g \wedge (x, y) = (x, y + g)$$

for all $g \in \mathbb{R}$ and all $(x, y) \in \mathbb{R}^2$ is an action of the group $(\mathbb{R}, +)$ on the set $\mathbb{R}^2$.

(a) Find all the orbits of this group action. Describe them geometrically, and sketch a diagram to show how they partition the plane.

(b) Show that under this group action the stabiliser of each point in $\mathbb{R}^2$ is the trivial subgroup of $(\mathbb{R}, +)$.

Additional Exercise E71

Let $G = \left\{ \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} : a \in \mathbb{R} \right\}$.

Then $(G, \times)$ is a group and the mapping effect $\wedge$ defined by

$$\begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \wedge (x, y) = \big((1+a)x + ay,\ -ax + (1-a)y\big)$$

for all $\begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$ is an action of the group $(G, \times)$ on the set $\mathbb{R}^2$. (You may assume these facts.)

(a) Find all the orbits of this group action. Describe them geometrically, and sketch a diagram to show how they partition the plane.

(b) Find the stabiliser of each of the following points.

(i) $(0, 0)$ (ii) $(1, 0)$ (iii) $(1, -1)$

(The proof that $(G, \times)$ is a group is similar to the solution to Additional Exercise E1(d). The fact that $\wedge$ is a group action then follows from Theorem E61 in Unit E4 and the result of Additional Exercise E68.)

Additional Exercise E72

Let $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in \mathbb{R}^* \right\}$.

Then $(G, \times)$ is a group and the mapping effect $\wedge$ defined by

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \wedge (x, y) = (ax, by)$$

for all $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$ is an action of the group $(G, \times)$ on the set $\mathbb{R}^2$. (You may assume these facts.)

Find all the orbits of this group action. Describe them geometrically, and sketch a diagram to show how they partition the plane.

(It is straightforward to show that $(G, \times)$ is a subgroup of $\mathrm{GL}(2)$ using the subgroup test, Theorem B24. The fact that $\wedge$ is a group action then follows from Theorem E61 in Unit E4 and the result of Additional Exercise E68.)

Additional Exercise E73

Consider the group $(\mathbb{R}^*, \times)$ and the set

$$X = \{(x, y) : x, y > 0\}.$$

(That is, X is the set of all points in the first quadrant of the plane.)

The solution to Additional Exercise E66 shows that the mapping effect $\wedge$ defined by

$$g \wedge (x, y) = (x^g, y)$$

for all $g \in \mathbb{R}^*$ and all $(x, y) \in X$ is an action of the group $(\mathbb{R}^*, \times)$ on the set X.

(a) Find all the orbits of this group action. Describe them geometrically, and sketch a diagram to show how they partition X (the first quadrant of the plane).

(b) Find the stabiliser of each point in X.

Additional Exercise E74

Consider the group S_4 and the set

$$X = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}\}.$$

Let $\wedge$ be the action of S_4 on X given by

$$g \wedge \{i, j\} = \{g(i), g(j)\}$$

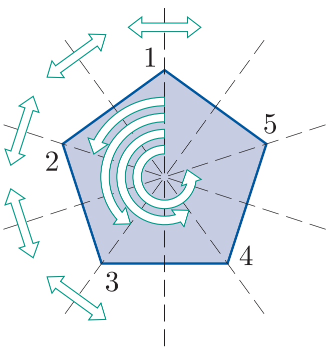
for all $g \in S_4$ and all $\{i, j\} \in X$. (This was shown to be a group action in the solution to Additional Exercise E64.)

(a) Show that this group action has only one orbit.

(b) Determine $\operatorname{Stab}\{i, j\}$ for each $\{i, j\} \in X$.

Additional Exercise E75

The non-identity symmetries of the regular pentagon are shown below.



Let $S(\pentagon)$ be represented as a group of permutations using the vertex labels shown above; then

$$\begin{aligned} S(\pentagon) = \{&e,\ (1\ 2\ 3\ 4\ 5),\ (1\ 3\ 5\ 2\ 4),\\ &(1\ 4\ 2\ 5\ 3),\ (1\ 5\ 4\ 3\ 2),\\ &(1\ 2)(3\ 5),\ (1\ 3)(4\ 5),\ (1\ 4)(2\ 3),\\ &(1\ 5)(2\ 4),\ (2\ 5)(3\ 4)\}. \end{aligned}$$

Let X be the set whose elements are the ten unordered pairs of symbols from $\{1, 2, 3, 4, 5\}$; that is,

$$\begin{aligned} X = \{&\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\},\\ &\{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}. \end{aligned}$$

Then the obvious mapping effect $\wedge$ of $S(\pentagon)$ on X, given by

$$g \wedge \{i, j\} = \{g(i), g(j)\}$$

for all $g \in S(\pentagon)$ and all $\{i, j\} \in X$, is a group action. (You may assume this.)

(a) Find the orbits of this group action.

(b) Determine $\operatorname{Stab}\{1, 2\}$ and $\operatorname{Stab}\{1, 3\}$.

Additional Exercise E76

The following table shows the effect of each element of a group $G = \{e, a, b, c, w, x, y, z\}$ on the set $X = \{1, 2, 3, 4, 5, 6, 7\}$ under an action of G on X. (You may assume that this is a group action.) Here i denotes the identity permutation of X.

Element g	Permutation
e	i
a	$(1\ 3\ 5\ 7)(2\ 4)$
b	$(1\ 5)(3\ 7)$
c	$(1\ 7\ 5\ 3)(2\ 4)$
w	$(1\ 3)(2\ 4)(5\ 7)$
x	$(1\ 5)$
y	$(1\ 7)(2\ 4)(3\ 5)$
z	$(3\ 7)$

(a) Find all the orbits of this group action.

(b) Write down the stabiliser of each element of X.

(c) For each $x \in X$, verify that

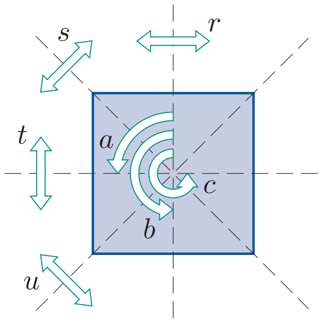
$$|\operatorname{Orb} x| \times |\operatorname{Stab} x| = |G|.$$

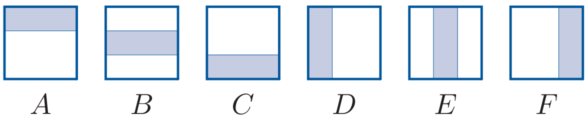
Section 3

Additional Exercise E77

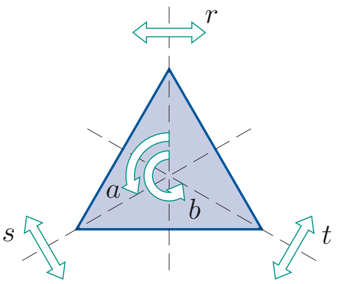
For each of the following group actions, verify the Orbit–Stabiliser Theorem for each element x of the set on which the group acts.

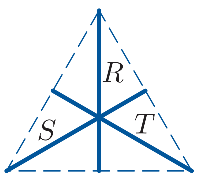
(a) The action of the group $S(\square)$ on the set $\{A, B, C, D, E, F\}$ of modified squares shown below. (This was shown to be a group action in the solution to Additional Exercise E65, and the orbits and stabilisers under this group action were found in the solution to Additional Exercise E69.)





(b) The action of the group $S(\triangle)$ on the lines of symmetry of the triangle, shown below.





(c) The natural action of the symmetric group S_4 on the set $X = \{1, 2, 3, 4\}$ of symbols.

Additional Exercise E78

Let G be a group, and let $\wedge$ be defined by

$$g \wedge x = g^{-1}xg$$

for all $g, x \in G$. Show that $\wedge$ is *not* a group action of G on itself.

(Proposition E68 in Unit E4 states that if G is a group and $\wedge$ is given by

$$g \wedge x = gxg^{-1}$$

for all $g, x \in G$, then $\wedge$ is a group action of G on itself.)

Additional Exercise E79

Let G be a group, let H be a subgroup of G and let X be the set of all left cosets of H in G. Let $\wedge$ be defined by

$$g \wedge xH = (gx)H$$

for all $g \in G$ and all $xH \in X$.

(a) Show that $\wedge$ is a group action of G on X.

(b) Determine $\operatorname{Orb} H$ and $\operatorname{Stab} H$ for this action.

(c) Which result can you deduce by applying the Orbit–Stabiliser Theorem (Theorem E64 in Unit E4) to your answers to part (b)?

Additional Exercise E80

The symmetry group of the cube, $S(\text{cube})$, has order 48.

(a) The group $S(\text{cube})$ has a natural action on the set of six faces of the cube. Explain why this group action has only one orbit, and hence show that for each face of the cube there are eight symmetries of the cube that fix the face. (Note that a face can be fixed without each point on the face being fixed.)

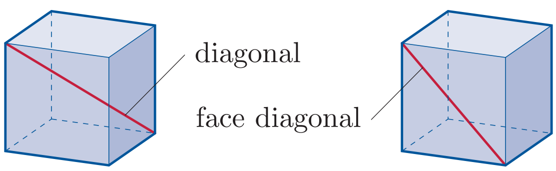
(b) For each of the following features of the cube, use a similar method to determine how many symmetries of the cube fix the feature.

(i) A vertex.

(ii) An edge.

(iii) A diagonal (illustrated below).

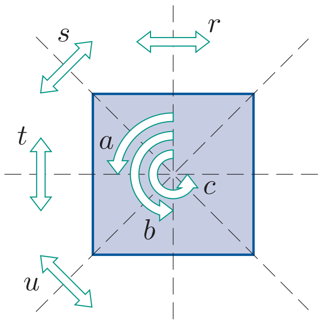
(iv) A face diagonal (illustrated below).



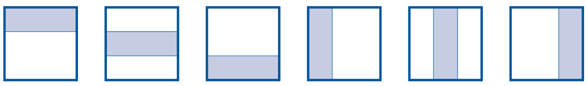
Section 4

Additional Exercise E81

The non-identity symmetries of the square are shown below.



Consider the action of the group $S(\square)$ on the set $\{A, B, C, D, E, F\}$ of modified squares shown below.



A B C D E F

(This was shown to be a group action in the solution to Additional Exercise E65.)

Write down the fixed set of each element of $S(\square)$ under this group action.

Additional Exercise E82

Let

$$G = \left\{ \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} : a \in \mathbb{R} \right\}.$$

Consider the action of the group $(G, \times)$ on the set $\mathbb{R}^2$ defined by

$$\begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \wedge (x, y)$$
$$= ((1+a)x + ay, -ax + (1-a)y)$$

for all $\begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G$ and all $(x, y) \in \mathbb{R}^2$.

(This is the same group action as in Additional Exercise E71.)

Find Fix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ under this group action, and describe it geometrically.

Additional Exercise E83

Consider the group $(\mathbb{R}^*, \times)$ and the set

$$X = \{(x, y) : x, y > 0\}.$$

(That is, X is the set of all points in the first quadrant of the plane.)

The solution to Additional Exercise E66 shows that the mapping effect $\wedge$ defined by

$$g \wedge (x, y) = (x^g, y)$$

for all $g \in \mathbb{R}^*$ and all $(x, y) \in X$ is an action of the group $(\mathbb{R}^*, \times)$ on the set X.

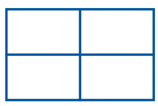
(a) Find an expression for the fixed set of a general element of the group $(\mathbb{R}^*, \times)$ under this group action.

(b) Find the fixed set of each of the following elements of $(\mathbb{R}^*, \times)$ under the group action. Describe each fixed set geometrically.

(i) 1 (ii) 2

Additional Exercise E84

(a) Use the Counting Theorem to determine how many different rectangular blankets similar to the one illustrated below can be made if each quadrant is to be coloured with one of two colours, and we regard two blankets as the same if one can be rotated or turned over to give the other.

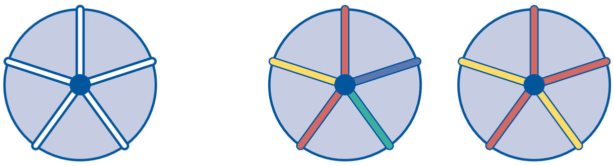


(b) Confirm your answer to part (a) by drawing the possibilities.

(c) Determine how many different coloured blankets can be made if each quadrant is to be coloured with one of three colours, rather than two.

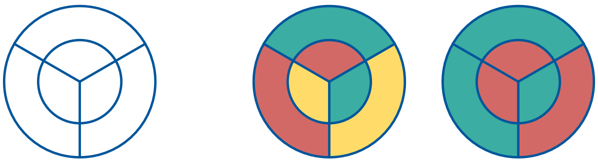
Additional Exercise E85

A playground roundabout, shown from above on the left below, has five painted rails, each of which can be blue, yellow, red or green. Two examples are shown on the right below. Use the Counting Theorem to determine how many different coloured roundabouts can be made if we regard two of them as the same when a rotation takes one to the other.



Additional Exercise E86

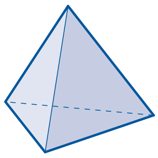
(a) Plastic pieces are used to construct coloured discs of the form shown on the left below. Two examples are shown on the right. The plastic pieces are available in three different colours. Use the Counting Theorem to determine how many different such coloured discs can be made if we regard two of them as the same when one can be rotated or turned over to give the other.



(b) Determine how many different such coloured discs can be made if the plastic pieces are available in only two colours rather than three.

Additional Exercise E87

Use the Counting Theorem to determine how many different regular tetrahedrons there are with each face painted blue, yellow, red or green, if we regard two of them as the same when a rotation takes one to the other.



(The rotational symmetries of the tetrahedron are described in the solution to Worked Exercise B14 in Subsection 5.3 of Unit B1.)

Solutions to additional exercises for Unit E4

Solution to Additional Exercise E64

We show that the group action axioms hold.

GA1 Let $g \in S_4$ and let $\{i, j\} \in X$. Then

$$g \wedge \{i, j\} = \{g(i), g(j)\}.$$

Now i and j are different symbols and g is a permutation, so $g(i)$ and $g(j)$ are also different symbols. Hence $\{g(i), g(j)\} \in X$. Thus axiom GA1 holds.

GA2 For each $\{i, j\} \in X$,

$$e \wedge \{i, j\} = \{e(i), e(j)\} = \{i, j\}.$$

Thus axiom GA2 holds.

GA3 Let $g, h \in S_4$ and let $\{i, j\} \in X$. Then

$$\begin{aligned} g \wedge (h \wedge \{i, j\}) &= g \wedge \{h(i), h(j)\} \\ &= \{g(h(i)),\ g(h(j))\} \\ &= \{(g \circ h)(i),\ (g \circ h)(j)\} \\ &= (g \circ h) \wedge \{i, j\}. \end{aligned}$$

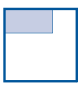
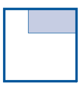
Thus axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

Solution to Additional Exercise E65

By Theorem E59, in each case $\wedge$ is a group action if and only if axiom GA1 holds, that is, if and only if every element of $S(\square)$ maps each figure in the set X to another figure in X.

(a) The element r of $S(\square)$ maps

 to .

The first figure here is an element of X but the second figure is not. So axiom GA1 does not hold. Hence, by Theorem E59, $\wedge$ is not a group action.

(b) We can see by inspection that every symmetry in $S(\square)$ maps each figure in X to another figure in X. So axiom GA1 holds. Hence, by Theorem E59, $\wedge$ is a group action.

Solution to Additional Exercise E66

We show that the group action axioms hold.

GA1 Let $g \in \mathbb{R}^*$ and let $(x, y) \in X$. Then

$$g \wedge (x, y) = (x^g, y).$$

Now $x > 0$ so $x^g > 0$. Hence $(x^g, y) \in X$. Thus axiom GA1 holds.

GA2 The identity element of the group $(\mathbb{R}^*, \times)$ is 1.

Let $(x, y) \in X$. Then

$$1 \wedge (x, y) = (x^1, y) = (x, y).$$

Thus axiom GA2 holds.

GA3 Let $g, h \in \mathbb{R}^*$ and let $(x, y) \in X$. We have to show that

$$g \wedge (h \wedge (x, y)) = (g \times h) \wedge (x, y).$$

Now

$$\begin{aligned} g \wedge (h \wedge (x, y)) &= g \wedge (x^h, y) \\ &= \big((x^h)^g, y\big) \\ &= (x^{gh}, y) \end{aligned}$$

and

$$(g \times h) \wedge (x, y) = (x^{g \times h}, y) = (x^{gh}, y).$$

The two expressions obtained are the same, so axiom GA3 holds.

Since the three group action axioms hold, $\wedge$ is a group action.

Solution to Additional Exercise E67

In each case we show that one of the group action axioms does not hold.

(a) The mapping effect $\wedge$ does not satisfy axiom GA2 (identity). The identity element of the group $(\mathbb{R}^+, \times)$ is 1, and, for example, $(2, 3) \in X$, but

$$1 \wedge (2, 3) = (1^2, 3) = (1, 3) \neq (2, 3).$$

Hence $\wedge$ is not a group action.

(b) The mapping effect $\wedge$ does not satisfy axiom GA3 (composition). For $\wedge$ to satisfy this axiom, the equation

$$g \wedge (h \wedge (x, y)) = (g \times h) \wedge (x, y)$$

would have to hold for all $g, h \in \mathbb{R}^+$ and all $(x, y) \in X$. However, for example, $2 \in \mathbb{R}^+$ and $(1, 1) \in X$, but

$$\begin{aligned} 2 \wedge (2 \wedge (1, 1)) &= 2 \wedge (2 \times 1^2, 1) \\ &= 2 \wedge (2, 1) \\ &= (2 \times 2^2, 1) \\ &= (8, 1) \end{aligned}$$

whereas

$$\begin{aligned} (2 \times 2) \wedge (1, 1) &= 4 \wedge (1, 1) \\ &= (4 \times 1^4, 1) \\ &(4, 1). \end{aligned}$$

Hence $\wedge$ is not a group action.

Solution to Additional Exercise E68

We show that the group action axioms hold.

GA1 Let $h \in H$ and let $x \in X$. Then

$$\begin{aligned} h \wedge_H x &= h \wedge x \\ &\in X \quad \text{(by axiom GA1 for } \wedge \text{, since } h \in G\text{)}. \end{aligned}$$

Thus axiom GA1 holds for $\wedge_H$.

GA2 Let the identity element of $(H, \circ)$ be e. Then the identity element of G is also e. Let $x \in X$. Then

$$\begin{aligned} e \wedge_H x &= e \wedge x \\ &= x \quad \text{(by axiom GA2 for } \wedge\text{)}. \end{aligned}$$

Thus axiom GA2 holds for $\wedge_H$.

GA3 Let $h_1, h_2 \in H$ and let $x \in X$. We have to show that

$$h_1 \wedge_H (h_2 \wedge_H x) = (h_1 \circ h_2) \wedge_H x.$$

Now

$$\begin{aligned} h_1 \wedge_H (h_2 \wedge_H x) &= h_1 \wedge (h_2 \wedge x) \\ &= (h_1 \circ h_2) \wedge x \\ &\qquad \text{(by axiom GA3 for } \wedge \text{, since } h_1, h_2 \in G\text{)} \\ &= (h_1 \circ h_2) \wedge_H x. \end{aligned}$$

Thus axiom GA3 holds for $\wedge_H$.

Since the three group action axioms hold, $\wedge_H$ is a group action.

Solution to Additional Exercise E69

(a) The orbits are

$$\{A, C, D, F\}, \quad \{B, E\}.$$

(b) The stabilisers are

$$\begin{aligned} &\operatorname{Stab} A = \{e, r\}, && \operatorname{Stab} D = \{e, t\}, \\ &\operatorname{Stab} B = \{e, b, r, t\}, && \operatorname{Stab} E = \{e, b, r, t\}, \\ &\operatorname{Stab} C = \{e, r\}, && \operatorname{Stab} F = \{e, t\}. \end{aligned}$$

Solution to Additional Exercise E70

(a) For any point $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} \operatorname{Orb}(x, y) &= \{g \wedge (x, y) : g \in \mathbb{R}\} \\ &= \{(x, y + g) : g \in \mathbb{R}\}. \end{aligned}$$

As an example, let us find $\operatorname{Orb}(1, 1)$. We have

$$\operatorname{Orb}(1, 1) = \{(1, 1 + g) : g \in \mathbb{R}\}.$$

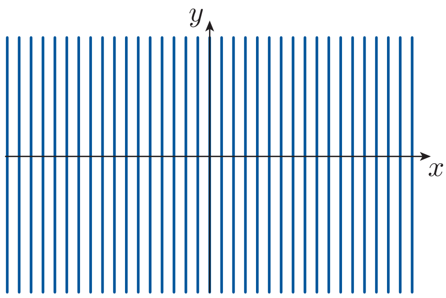
As g runs through all the values in $\mathbb{R}$, the point $(1, 1 + g)$ moves through all the points with x-coordinate 1. So $\operatorname{Orb}(1, 1)$ is the set of points on the vertical line through $(1, 1)$.

In general, as found above, we have

$$\operatorname{Orb}(x, y) = \{(x, y + g) : g \in \mathbb{R}\}.$$

As g runs through all the values in $\mathbb{R}$, the point $(x, y + g)$ moves through all the points with x-coordinate x. So $\operatorname{Orb}(x, y)$ is the set of points on the vertical line through (x, y).

Thus the orbits of this group action are the vertical lines, as sketched below. These partition the plane, as expected.



(b) For any point $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned} \operatorname{Stab}(x, y) &= \{g \in \mathbb{R} : g \wedge (x, y) = (x, y)\} \\ &= \{g \in \mathbb{R} : (x, y + g) = (x, y)\} \\ &= \{g \in \mathbb{R} : g = 0\} \\ &= \{0\}. \end{aligned}$$

Thus the stabiliser of each point in $\mathbb{R}^2$ is the trivial subgroup of $(\mathbb{R}, +)$.

Solution to Additional Exercise E71

(a) For any element $(x,y) \in \mathbb{R}^2$,

$$\begin{aligned}
&\operatorname{Orb}(x,y)\\
&= \left\{ \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \wedge (x,y) : \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G \right\}\\
&= \left\{ \big((1+a)x+ay,\ -ax+(1-a)y\big) : a \in \mathbb{R} \right\}.
\end{aligned}$$

(The next part of the solution is not an essential part of it, but it demonstrates how we might obtain an idea of how the plane splits up into orbits.)

Let us start by finding the orbits of a few points. We have

$$\begin{aligned}
&\operatorname{Orb}(0,0)\\
&= \left\{ \big((1+a)\times 0 + a\times 0,\ -a\times 0+(1-a)\times 0\big) : a \in \mathbb{R} \right\}\\
&= \{(0,0)\}.
\end{aligned}$$

So $\operatorname{Orb}(0,0)$ contains the point $(0,0)$ only.

Also

$$\begin{aligned}
&\operatorname{Orb}(1,0)\\
&= \left\{ \big((1+a)\times 1 + a\times 0,\ -a\times 1+(1-a)\times 0\big) : a \in \mathbb{R} \right\}\\
&= \{(1+a,-a) : a \in \mathbb{R}\}.
\end{aligned}$$

So $\operatorname{Orb}(1,0)$ is a straight line with gradient -1 (specifically, the line $y=-x+1$).

Similarly

$$\begin{aligned}
&\operatorname{Orb}(1,1)\\
&= \left\{ \big((1+a)\times 1 + a\times 1,\ -a\times 1+(1-a)\times 1\big) : a \in \mathbb{R} \right\}\\
&= \{(1+2a,-2a+1) : a \in \mathbb{R}\}.
\end{aligned}$$

So $\operatorname{Orb}(1,1)$ is also a straight line with gradient -1.

It looks as through the orbits may be the straight lines with gradient -1, except that we know that the line $y=-x$ is not an orbit, because we know that the origin lies in an orbit containing itself alone. Let us try finding the orbit of another point on the line $y=-x$. We have

$$\begin{aligned}
&\operatorname{Orb}(1,-1)\\
&= \left\{ \big((1+a)\times 1 + a(-1),\ -a\times 1+(1-a)(-1)\big) : a \in \mathbb{R} \right\}\\
&= \{(1,-1)\}.
\end{aligned}$$

So $\operatorname{Orb}(1,-1)$ contains the point $(1,-1)$ only.

It looks as though the orbits may be the straight lines with gradient -1 except for the line $y=-x$, together with the individual points on the line $y=-x$. We now try to confirm this algebraically.

For any point $(x,-x)$, that is, any point on the line $y=-x$, we have

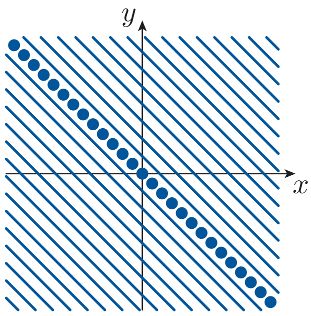
$$\begin{aligned}
&\operatorname{Orb}(x,-x)\\
&= \left\{ \big((1+a)x + a(-x),\ -ax+(1-a)(-x)\big) : a \in \mathbb{R} \right\}\\
&= \{(x,-x)\}.
\end{aligned}$$

For any point (x,y) with $y \neq -x$, that is, any point not on the line $y=-x$, we have

$$\begin{aligned}
&\operatorname{Orb}(x,y)\\
&= \left\{ \big((1+a)x+ay,\ -ax+(1-a)y\big) : a \in \mathbb{R} \right\}\\
&= \left\{ \big(x+a(x+y),\ y-a(x+y)\big) : a \in \mathbb{R} \right\}.
\end{aligned}$$

Since $x+y \neq 0$, as a runs through all the values in $\mathbb{R}$, the point $\big(x+a(x+y),\ y-a(x+y)\big)$ moves through all the points on the line through the point (x,y) with gradient -1. So $\operatorname{Orb}(x,y)$ is this line.

Thus the orbits are indeed as described above. They are illustrated below.



(b) For any point $(x,y) \in \mathbb{R}^2$,

$$\begin{aligned}
&\operatorname{Stab}(x,y)\\
&= \left\{ \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G : \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \wedge (x,y) = (x,y) \right\}\\
&= \left\{ \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G : \big((1+a)x+ay,\ -ax+(1-a)y\big) = (x,y) \right\}\\
&= \left\{ \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G : \big(x+ax+ay,\ -ax+y-ay\big) = (x,y) \right\}\\
&= \left\{ \begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G : a(x+y)=0 \right\}.
\end{aligned}$$

This gives the following.

(i) Stab$(0,0)$

$$= \left\{\begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G : a \times 0 = 0\right\}$$
$$= G.$$

(ii) Stab$(1,0)$

$$= \left\{\begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G : a \times 1 = 0\right\}$$
$$= \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

(iii) Stab$(1,-1)$

$$= \left\{\begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix} \in G : a \times 0 = 0\right\}$$
$$= G.$$

Solution to Additional Exercise E72

For any element $(x,y) \in \mathbb{R}^2$,

$$\begin{aligned}\mathrm{Orb}(x,y) &= \left\{\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \wedge (x,y) : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in G\right\} \\ &= \{(ax, by) : a, b \in \mathbb{R}^*\}.\end{aligned}$$

Let us find the orbits of a few points.

We have

$$\begin{aligned}\mathrm{Orb}(0,0) &= \{(a \times 0, b \times 0) : a, b \in \mathbb{R}^*\} \\ &= \{(0,0)\}.\end{aligned}$$

So Orb$(0,0)$ contains the point $(0,0)$ only.

Also

$$\begin{aligned}\mathrm{Orb}(1,0) &= \{(a \times 1, b \times 0) : a, b \in \mathbb{R}^*\} \\ &= \{(a,0) : a \in \mathbb{R}^*\}.\end{aligned}$$

So Orb$(1,0)$ is the x-axis excluding the origin.

Similarly

$$\begin{aligned}\mathrm{Orb}(0,1) &= \{(a \times 0, b \times 1) : a, b \in \mathbb{R}^*\} \\ &= \{(0,b) : b \in \mathbb{R}^*\}.\end{aligned}$$

So Orb$(0,1)$ is the y-axis excluding the origin.

Also

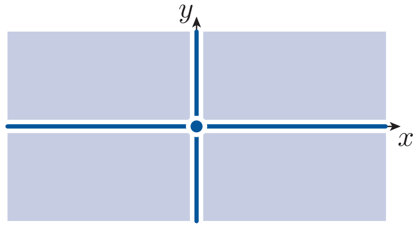
$$\begin{aligned}\mathrm{Orb}(1,1) &= \{(a \times 1, b \times 1) : a, b \in \mathbb{R}^*\} \\ &= \{(a,b) : a, b \in \mathbb{R}^*\}.\end{aligned}$$

So Orb$(1,1)$ is the whole plane excluding the x-axis and y-axis.

Thus this group action has four orbits, as follows.

- The origin.
- The x-axis excluding the origin.
- The y-axis excluding the origin.
- The rest of the plane.

These orbits are illustrated below. The lines and the area continue on the other side of the origin.



(This exercise is similar to Exercise E152 in Subsection 2.2 of Unit E4, but the definition of the group G is slightly different: it involves $a, b \in \mathbb{R}^*$ rather than $a, b \in \mathbb{R}^+$. This leads to different orbits.)

Solution to Additional Exercise E73

(a) For any point $(x,y) \in X$,

$$\begin{aligned}\mathrm{Orb}(x,y) &= \{g \wedge (x,y) : g \in \mathbb{R}^*\} \\ &= \{(x^g, y) : g \in \mathbb{R}^*\}.\end{aligned}$$

As an example, let us find Orb$(1,1)$. We have

$$\begin{aligned}\mathrm{Orb}(1,1) &= \{(1^g, 1) : g \in \mathbb{R}^*\} \\ &= \{(1,1)\}.\end{aligned}$$

Similarly, for any point $(1,y)$ with $y > 0$, that is, any first-quadrant point that lies on the line $x = 1$, we have

$$\begin{aligned}\mathrm{Orb}(1,y) &= \{g \wedge (1,y) : g \in \mathbb{R}^*\} \\ &= \{(1^g, y) : g \in \mathbb{R}^*\} \\ &= \{(1,y)\}.\end{aligned}$$

Thus any point in X that is on the line $x = 1$ lies in an orbit that contains itself only.

Now let us find Orb$(2,3)$, for example. We have

$$\mathrm{Orb}(2,3) = \{(2^g, 3) : g \in \mathbb{R}^*\}.$$

As g runs through all the values in $\mathbb{R}^*$, the point $(2^g, 3)$ moves through all the points of the form $(r, 3)$ with $r > 0$ and $r \neq 1$. That is, the point $(2^g, 3)$ moves through all the first-quadrant points that lie on the same horizontal line as $(2,3)$ except the point $(1,3)$. So Orb$(2,3)$ is the horizontal 'half-line' through $(2,3)$ excluding the point $(1,3)$.

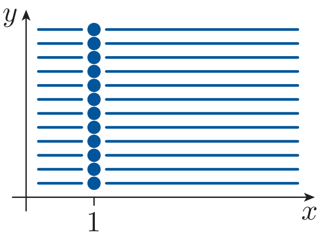
In general, for any point (x, y) with $x, y > 0$ and $x \neq 1$, that is, any first-quadrant point that does not lie on the line $x = 1$, we have

$$\text{Orb}(x, y) = \{(x^g, y) : g \in \mathbb{R}^*\}.$$

As g runs through all the values in $\mathbb{R}^*$, the point (x^g, y) moves through all the points of the form (r, y) with $r > 0$ and $r \neq 1$. That is, the point (x^g, y) moves through all the first-quadrant points that lie on the same horizontal line as (x, y) except the point $(1, y)$. So $\text{Orb}(x, y)$ is the horizontal half-line through (x, y) excluding the point $(1, y)$.

We have now found all the orbits. They are the individual first-quadrant points on the line $x = 1$ and the horizontal half-lines in the first quadrant excluding the point with x-coordinate 1 in each such half-line.

They are illustrated below. Each orbit that is a half-line continues on the other side of the line $x = 1$. These orbits partition the first quadrant of the plane, as expected.



(b) For any point $(x, y) \in X$,

$$\begin{aligned}\text{Stab}(x, y) &= \{g \in \mathbb{R}^* : g \wedge (x, y) = (x, y)\} \\ &= \{g \in \mathbb{R}^* : (x^g, y) = (x, y)\} \\ &= \{g \in \mathbb{R} : x^g = x\}.\end{aligned}$$

Hence for any point $(x, y) \in X$ with $x \neq 1$ (that is, any first-quadrant point not on the line $x = 1$),

$$\begin{aligned}\text{Stab}(x, y) &= \{g \in \mathbb{R}^* : x^g = x\} \\ &= \{g \in \mathbb{R}^* : g = 1\} \quad (\text{since } x \neq 1) \\ &= \{1\}.\end{aligned}$$

Also, for any point $(1, y) \in X$ (that is, any first-quadrant point on the line $x = 1$),

$$\begin{aligned}\text{Stab}(1, y) &= \{g \in \mathbb{R}^* : 1^g = 1\} \\ &= \mathbb{R}^*.\end{aligned}$$

In summary, the stabiliser of any point in X on the line $x = 1$ is the whole group $\mathbb{R}^*$, and the stabiliser of any other point in X is the trivial subgroup $\{1\}$.

Solution to Additional Exercise E74

(a) The orbit of $\{1, 2\}$ contains the following elements:

$$\begin{aligned} e \wedge \{1,2\} &= \{1,2\}, \\ (2\ 3) \wedge \{1,2\} &= \{1,3\}, \\ (2\ 4) \wedge \{1,2\} &= \{1,4\}, \\ (1\ 3) \wedge \{1,2\} &= \{2,3\}, \\ (1\ 4) \wedge \{1,2\} &= \{2,4\}, \\ (1\ 3)(2\ 4) \wedge \{1,2\} &= \{3,4\}. \end{aligned}$$

These are the six elements of X, so $\text{Orb}\{1, 2\} = X$. Thus there is just one orbit.

(b) For each $\{i, j\} \in X$,

$$\begin{aligned}\text{Stab}\{i, j\} &= \{g \in S_4 : g \wedge \{i, j\} = \{i, j\}\} \\ &= \{g \in S_4 : \{g(i), g(j)\} = \{i, j\}\}.\end{aligned}$$

So $\text{Stab}\{i, j\}$ consists of the elements g in S_4 that either fix each of i and j (so $g(i) = i$ and $g(j) = j$), or transpose i and j (so $g(i) = j$ and $g(j) = i$). For example,

$$\text{Stab}\{1, 2\} = \{\underbrace{e,\ (3\ 4)}_{\substack{\text{These}\\ \text{elements}\\ \text{fix}\\ \text{1 and 2.}}},\ \underbrace{(1\ 2),\ (1\ 2)(3\ 4)}_{\substack{\text{These}\\ \text{elements}\\ \text{transpose}\\ \text{1 and 2.}}}\}.$$

Thus the stabilisers of the elements of X are

$$\begin{aligned}\text{Stab}\{1, 2\} &= \{e,\ (3\ 4),\ (1\ 2),\ (1\ 2)(3\ 4)\}, \\ \text{Stab}\{1, 3\} &= \{e,\ (2\ 4),\ (1\ 3),\ (1\ 3)(2\ 4)\}, \\ \text{Stab}\{1, 4\} &= \{e,\ (2\ 3),\ (1\ 4),\ (1\ 4)(2\ 3)\}, \\ \text{Stab}\{2, 3\} &= \{e,\ (1\ 4),\ (2\ 3),\ (1\ 4)(2\ 3)\}, \\ \text{Stab}\{2, 4\} &= \{e,\ (1\ 3),\ (2\ 4),\ (1\ 3)(2\ 4)\}, \\ \text{Stab}\{3, 4\} &= \{e,\ (1\ 2),\ (3\ 4),\ (1\ 2)(3\ 4)\}.\end{aligned}$$

Solution to Additional Exercise E75

(a) We start by choosing any element of X, say $\{1, 2\}$, and finding its orbit. We have

$$\begin{aligned} e \wedge \{1,2\} &= \{1,2\}, \\ (2\ 5)(3\ 4) \wedge \{1,2\} &= \{1,5\}, \\ (1\ 3)(4\ 5) \wedge \{1,2\} &= \{2,3\}, \\ (1\ 5)(2\ 4) \wedge \{1,2\} &= \{4,5\}, \\ (1\ 2)(3\ 5) \wedge \{1,2\} &= \{1,2\}, \\ (1\ 4)(2\ 3) \wedge \{1,2\} &= \{3,4\}, \\ (1\ 2\ 3\ 4\ 5) \wedge \{1,2\} &= \{2,3\}, \\ (1\ 3\ 5\ 2\ 4) \wedge \{1,2\} &= \{3,4\}, \\ (1\ 4\ 2\ 5\ 3) \wedge \{1,2\} &= \{4,5\}, \\ (1\ 5\ 4\ 3\ 2) \wedge \{1,2\} &= \{1,5\}. \end{aligned}$$

So

$$\text{Orb}\{1,2\} = \{\{1,2\}, \{2,3\}, \{3,4\}, \{4,5\}, \{1,5\}\}.$$

Next we choose an element not in the orbit already found, say $\{1,3\}$, and find its orbit. We have

$$\begin{aligned} e \wedge \{1,3\} &= \{1,3\},\\ (1\ 2\ 3\ 4\ 5) \wedge \{1,3\} &= \{2,4\},\\ (1\ 3\ 5\ 2\ 4) \wedge \{1,3\} &= \{3,5\},\\ (1\ 4\ 2\ 5\ 3) \wedge \{1,3\} &= \{1,4\},\\ (1\ 5\ 4\ 3\ 2) \wedge \{1,3\} &= \{2,5\}. \end{aligned}$$

All the elements of X have now been assigned to orbits. Hence

$$\text{Orb}\{1,3\} = \{\{1,3\}, \{2,4\}, \{3,5\}, \{1,4\}, \{2,5\}\},$$

and the orbits of the action are the two orbits found above.

(b) We have

$$\begin{aligned} &\text{Stab}\{1,2\}\\ &= \{g \in S(\pentagon) : g \wedge \{1,2\} = \{1,2\}\}\\ &= \{g \in S(\pentagon) : \{g(1), g(2)\} = \{1,2\}\}\\ &= \{g \in S(\pentagon) : g \text{ fixes or transposes 1 and 2}\}. \end{aligned}$$

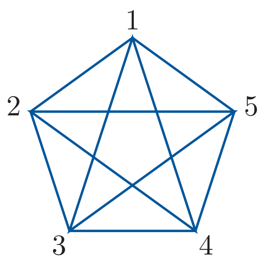
The only element of $S(\pentagon)$ that fixes both 1 and 2 is the identity element, and the only element that transposes 1 and 2 is $(1\ 2)(3\ 5)$. Hence

$$\text{Stab}\{1,2\} = \{e,\ (1\ 2)(3\ 5)\}.$$

Similarly, the elements of $\text{Stab}\{1,3\}$ are the elements of $S(\pentagon)$ that either fix or transpose 1 and 3, so

$$\text{Stab}\{1,3\} = \{e,\ (1\ 3)(4\ 5)\}.$$

(The group action in this question is the action of the group $S(\pentagon)$ on the set whose elements are the ten straight line segments in the figure below.
In the question each line segment is represented by an unordered pair of symbols from the set $\{1,2,3,4,5\}$, namely the labels of its endpoints.)



Solution to Additional Exercise E76

(a) From the given cycle forms we see that 1 is mapped by elements of G to 3, 5 and 7, but not to 2, 4 or 6. Thus

$$\text{Orb}\,1 = \{1,3,5,7\}.$$

Similarly,

$$\text{Orb}\,2 = \{2,4\}$$

and

$$\text{Orb}\,6 = \{6\}.$$

So the orbits of the action are

$$\{1,3,5,7\}, \quad \{2,4\}, \quad \{6\}.$$

(b) From the given cycle forms we see that 1 is fixed by e and z but not by any other element of G. Thus $\text{Stab}\,1 = \{e,z\}$. Finding the stabilisers of all the elements of G in a similar way gives

$$\begin{aligned} \text{Stab}\,1 &= \{e,z\},\\ \text{Stab}\,2 &= \{e,b,x,z\},\\ \text{Stab}\,3 &= \{e,x\},\\ \text{Stab}\,4 &= \{e,b,x,z\},\\ \text{Stab}\,5 &= \{e,z\},\\ \text{Stab}\,6 &= \{e,a,b,c,w,x,y,z\} = G,\\ \text{Stab}\,7 &= \{e,x\}. \end{aligned}$$

(c) From parts (a) and (b) we obtain the following table.

x	$\lvert\text{Orb}\,x\rvert$	$\lvert\text{Stab}\,x\rvert$	$\lvert\text{Orb}\,x\rvert \times \lvert\text{Stab}\,x\rvert$
1	4	2	8
2	2	4	8
3	4	2	8
4	2	4	8
5	4	2	8
6	1	8	8
7	4	2	8

In each case,

$$\lvert\text{Orb}\,x\rvert \times \lvert\text{Stab}\,x\rvert = \lvert G\rvert.$$

Solution to Additional Exercise E77

(a) The orbits of this group action are

$$\{A, C, D, F\}, \quad \{B, E\}.$$

Also,

$$\begin{aligned}
\operatorname{Stab} A &= \{e, r\},\\
\operatorname{Stab} B &= \{e, b, r, t\},\\
\operatorname{Stab} C &= \{e, r\},\\
\operatorname{Stab} D &= \{e, t\},\\
\operatorname{Stab} E &= \{e, b, r, t\},\\
\operatorname{Stab} F &= \{e, t\}.
\end{aligned}$$

So for each $x \in \{A, C, D, F\}$ we have

$$|\operatorname{Orb} x| \times |\operatorname{Stab} x| = 4 \times 2 = 8 = |S(\square)|,$$

and for each $x \in \{B, E\}$ we have

$$|\operatorname{Orb} x| \times |\operatorname{Stab} x| = 2 \times 4 = 8 = |S(\square)|.$$

(b) This group action has just one orbit, namely

$$\{R, S, T\}.$$

Also,

$$\begin{aligned}
\operatorname{Stab} R &= \{e, r\},\\
\operatorname{Stab} S &= \{e, s\},\\
\operatorname{Stab} T &= \{e, t\}.
\end{aligned}$$

Hence, for each line of symmetry x,

$$|\operatorname{Orb} x| \times |\operatorname{Stab} x| = 3 \times 2 = 6 = |S(\triangle)|.$$

(c) The elements of X form a single orbit, so $|\operatorname{Orb} x| = 4$ for each $x \in X$.

For each $x \in X$, $\operatorname{Stab} x$ consists of the elements of S_4 that fix x, and there are six such elements, corresponding to the six permutations of the other three symbols. For example,

$$\operatorname{Stab} 1 = \{e, (2\ 3), (2\ 4), (3\ 4), (2\ 3\ 4), (2\ 4\ 3)\}.$$

Hence $|\operatorname{Stab} x| = 6$ for each $x \in X$.

Therefore, for each $x \in X$,

$$|\operatorname{Orb} x| \times |\operatorname{Stab} x| = 4 \times 6 = 24 = |S_4|.$$

Solution to Additional Exercise E78

Axiom GA3 does not hold. If $g, h, x \in G$, then

$$\begin{aligned}
g \wedge (h \wedge x) &= g \wedge (h^{-1}xh)\\
&= g^{-1}(h^{-1}xh)g\\
&= (g^{-1}h^{-1})x(hg)\\
&= (hg)^{-1}x(hg)
\end{aligned}$$

but

$$(gh) \wedge x = (gh)^{-1}x(gh).$$

These two expressions are equal when $gh = hg$. This is not true in general, but it does hold when the group G is abelian.

As a particular counterexample to demonstrate that axiom GA3 does not always hold, consider the group S_4 and its elements $g = (1\ 2)$, $h = (1\ 3)$ and $x = (1\ 2\ 4)$. Then

$$g \circ h = (1\ 3\ 2) \quad \text{and} \quad h \circ g = (1\ 2\ 3),$$

so

$$\begin{aligned}
g \wedge (h \wedge x) &= (h \circ g)^{-1}x(h \circ g) \quad \text{(by the above)}\\
&= (1\ 2\ 3)^{-1}(1\ 2\ 4)(1\ 2\ 3)\\
&= (1\ 3\ 2)(1\ 2\ 4)(1\ 3\ 2)^{-1}\\
&= (3\ 1\ 4) \quad \text{(by the renaming method)}\\
&= (1\ 4\ 3)
\end{aligned}$$

but

$$\begin{aligned}
(g \circ h) \wedge x &= (g \circ h)^{-1}x(g \circ h)\\
&= (1\ 3\ 2)^{-1}(1\ 2\ 4)(1\ 3\ 2)\\
&= (1\ 2\ 3)(1\ 2\ 4)(1\ 2\ 3)^{-1}\\
&= (2\ 3\ 4) \quad \text{(by the renaming method).}
\end{aligned}$$

Thus

$$(h \circ g)^{-1}x(h \circ g) \neq (g \circ h)^{-1}x(g \circ h),$$

so axiom GA3 does not hold in this case.

Solution to Additional Exercise E79

(a) We show that the group action axioms hold.

GA1 Let $g \in G$ and let $xH \in X$. Then

$$g \wedge xH = (gx)H \in X.$$

Thus axiom GA1 holds.

GA2 Let e be the identity element of G and let $xH \in X$. Then

$$e \wedge xH = (ex)H = xH.$$

Thus axiom GA2 holds.

GA3 Let $g, h \in G$ and let $xH \in X$. Then

$$\begin{aligned}
g \wedge (h \wedge xH) &= g \wedge (hx)H\\
&= (ghx)H
\end{aligned}$$

and

$$(gh) \wedge xH = (ghx)H.$$

Thus

$$g \wedge (h \wedge xH) = (gh) \wedge xH,$$

so axiom GA3 holds.

Hence $\wedge$ is a group action.

(b) We have

$$\begin{aligned}\operatorname{Orb} H &= \{g \wedge H : g \in G\}\\ &= \{gH : g \in G\}\\ &= X,\end{aligned}$$

since $\{gH : g \in G\}$ is the set of all left cosets of H in G.

Also

$$\begin{aligned}\operatorname{Stab} H &= \{g \in G : g \wedge H = H\}\\ &= \{g \in G : gH = H\}\\ &= H,\end{aligned}$$

since a left coset gH is equal to H if and only if $g \in H$.

(c) If G is finite, then applying the Orbit–Stabiliser Theorem to the solutions to part (b) gives

$$|X| \times |H| = |G|.$$

That is, the number of left cosets of H in G times the order of H equals the order of G. From this result we can immediately deduce Lagrange's Theorem (thus obtaining it in yet another way).

Solution to Additional Exercise E80

(a) Any face of the cube can be mapped to any other face by, for example, a rotation of the cube. Hence the six faces form a single orbit.

Therefore, by the Orbit–Stabiliser Theorem (Theorem E64), the stabiliser of any face of the cube has order $48/6 = 8$. That is, there are eight symmetries of the cube that fix the face.

(b) **(i)** The eight vertices form a single orbit (since any vertex can be mapped to any other vertex by a symmetry of the cube). So for each vertex there are $48/8 = 6$ symmetries that fix the vertex.

(ii) The 12 edges form a single orbit (since any edge can be mapped to any other edge by a symmetry of the cube). So for each edge there are $48/12 = 4$ symmetries that fix the edge.

(iii) There are four diagonals, and these form a single orbit (since any diagonal can be mapped to any other diagonal by a symmetry of the cube). So for each diagonal there are $48/4 = 12$ symmetries that fix the diagonal.

(iv) Each of the six faces has two diagonals, so there are 12 face diagonals altogether. These form a single orbit because any face diagonal can be mapped to any other face diagonal by a symmetry of the cube. Thus for each face diagonal there are $48/12 = 4$ symmetries that fix the face diagonal.

Solution to Additional Exercise E81

The fixed sets are

$$\begin{aligned}&\operatorname{Fix} e = \{A, B, C, D, E, F\}, && \operatorname{Fix} a = \varnothing,\\ &\operatorname{Fix} b = \{B, E\}, && \operatorname{Fix} c = \varnothing,\\ &\operatorname{Fix} r = \{A, B, C, E\}, && \operatorname{Fix} s = \varnothing,\\ &\operatorname{Fix} t = \{B, D, E, F\}, && \operatorname{Fix} u = \varnothing.\end{aligned}$$

Solution to Additional Exercise E82

The matrix $\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ is of the form $\begin{pmatrix} 1+a & a \\ -a & 1-a \end{pmatrix}$ with $a = 1$.

Thus

$$\begin{aligned}&\operatorname{Fix}\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}\\ &= \left\{(x,y) \in \mathbb{R}^2 : \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix} \wedge (x,y) = (x,y)\right\}\\ &= \left\{(x,y) \in \mathbb{R}^2 : (2x+y, -x) = (x,y)\right\}\\ &= \left\{(x,y) \in \mathbb{R}^2 : y = -x\right\}.\end{aligned}$$

That is, the required fixed set is the line $y = -x$.

Solution to Additional Exercise E83

(a) For any number $g \in \mathbb{R}^*$,

$$\begin{aligned}\operatorname{Fix} g &= \{(x,y) \in X : g \wedge (x,y) = (x,y)\}\\ &= \{(x,y) \in X : (x^g, y) = (x,y)\}\\ &= \{(x,y) \in X : x^g = x\}.\end{aligned}$$

(b) **(i)** By part (a),

$$\begin{aligned}\operatorname{Fix} 1 &= \left\{(x,y) \in X : x^1 = x\right\}\\ &= X.\end{aligned}$$

So $\operatorname{Fix} 1$ is the whole set X, that is, the whole first quadrant.

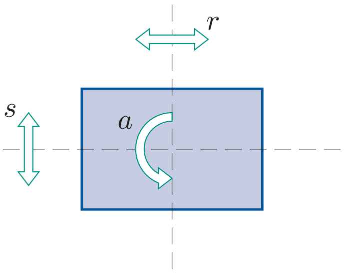
(ii) By part (a),

$$\begin{aligned}\operatorname{Fix} 2 &= \left\{(x,y) \in X : x^2 = x\right\}\\ &= \{(x,y) \in X : x = 1\} \quad (\text{since } x > 0)\\ &= \{(1,y) : y > 0\}.\end{aligned}$$

So $\operatorname{Fix} 2$ is the part of the line $x = 1$ that lies in the first quadrant.

Solution to Additional Exercise E84

(a) We want to regard two coloured blankets as the same if one can be rotated or reflected to give the other. So we consider the action of the group S(▭) (see below) on the set of all possible coloured blankets in fixed positions.



We can label the regions as shown below.

1	4
2	3

The sizes of the fixed sets for this group action are as given below.

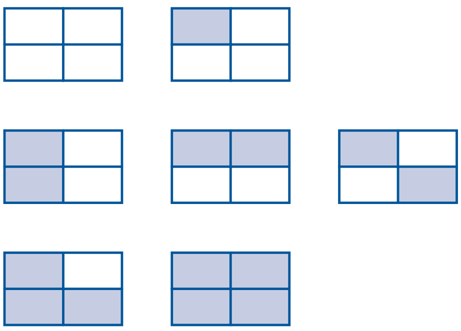
Symmetry g	Permutation	Number of cycles	$\lvert\text{Fix}\, g\rvert$
e	$(1)(2)(3)(4)$	4	2^4
a	$(1\ 3)(2\ 4)$	2	2^2
r	$(1\ 4)(2\ 3)$	2	2^2
s	$(1\ 2)(3\ 4)$	2	2^2

By the Counting Theorem, the number of orbits is

$$\begin{aligned}\tfrac{1}{4}\left(2^4 + 3 \times 2^2\right) &= \tfrac{1}{4} \times 2^2(2^2 + 3)\\ &= 7.\end{aligned}$$

Hence there are seven different coloured blankets.

(b) The seven possibilities are shown below.



(c) To determine the number of coloured blankets when three colours are available, we rework the calculation in the solution to part (a), changing the number of colours from 2 to 3 wherever it occurs. This gives the answer

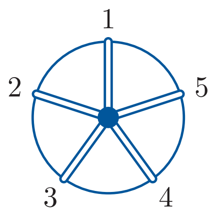
$$\begin{aligned}\tfrac{1}{4}\left(3^4 + 3 \times 3^2\right) &= \tfrac{1}{4} \times 3^3(3 + 1)\\ &= 27.\end{aligned}$$

Hence there are 27 different coloured blankets if three colours are available.

Solution to Additional Exercise E85

We want to regard two coloured roundabouts as the same if one can be rotated to give the other. So we consider the action of the group S^+(⬠) on the set of all possible coloured roundabouts in fixed positions.

We can label the rails as shown below.



The sizes of the fixed sets for this group action are as given below.

Rotation g through	Permutation	Number of cycles	$\lvert\text{Fix}\, g\rvert$
0	$(1)(2)(3)(4)(5)$	5	4^5
$2\pi/5$	$(1\ 2\ 3\ 4\ 5)$	1	4
$4\pi/5$	$(1\ 3\ 5\ 2\ 4)$	1	4
$6\pi/5$	$(1\ 4\ 2\ 5\ 3)$	1	4
$8\pi/5$	$(1\ 5\ 4\ 3\ 2)$	1	4

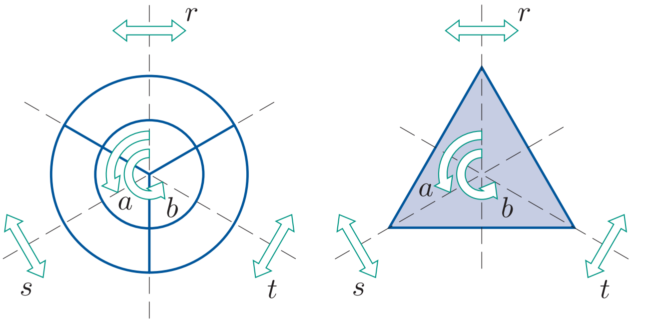
By the Counting Theorem, the number of orbits is

$$\begin{aligned}\tfrac{1}{5}\left(4^5 + 4 \times 4\right) &= \tfrac{1}{5} \times 4^2(4^3 + 1)\\ &= \tfrac{16}{5} \times 65\\ &= 16 \times 13\\ &= 4 \times 52\\ &= 208.\end{aligned}$$

Hence there are 208 different coloured roundabouts.

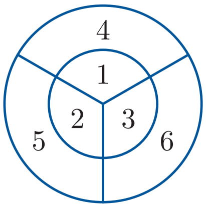
Solution to Additional Exercise E86

(a) The symmetry group of the modified disc to be coloured is $S(\triangle)$, as shown below.



We want to regard two coloured discs as the same if one can be rotated or reflected to give the other. So we consider the action of the group $S(\triangle)$ on the set of all possible coloured discs in fixed positions.

We can label the regions as shown below.



The sizes of the fixed sets for this group action are as given below.

Symmetry g	Permutation	Number of cycles	$\lvert\text{Fix}\, g\rvert$
e	$(1)(2)(3)(4)(5)(6)$	6	3^6
a	$(1\ 2\ 3)(4\ 5\ 6)$	2	3^2
b	$(1\ 3\ 2)(4\ 6\ 5)$	2	3^2
r	$(1)(2\ 3)(4)(5\ 6)$	4	3^4
s	$(1\ 3)(2)(4\ 6)(5)$	4	3^4
t	$(1\ 2)(3)(4\ 5)(6)$	4	3^4

By the Counting Theorem, the number of orbits is

$$\begin{aligned}\tfrac{1}{6}\left(3^6 + 2\times 3^2 + 3\times 3^4\right) &= \tfrac{1}{6}\times 3^2(3^4+2+3^3)\\ &= \tfrac{3}{2}(81+2+27)\\ &= \tfrac{3}{2}\times 110\\ &= 165.\end{aligned}$$

Hence there are 165 different coloured discs.

(b) To determine the number of coloured cubes when only two colours are available, we rework the calculation in the solution to part (a), changing the number of colours from 3 to 2 wherever it occurs. This gives the answer

$$\begin{aligned}\tfrac{1}{6}\left(2^6 + 2\times 2^2 + 3\times 2^4\right) &= \tfrac{1}{6}\times 2^3(2^3+1+3\times 2)\\ &= \tfrac{4}{3}(8+1+6)\\ &= \tfrac{4}{3}\times 15\\ &= 20.\end{aligned}$$

Hence there are 20 different coloured discs if only two colours are available.

Solution to Additional Exercise E87

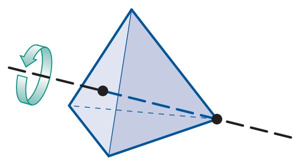
The solution given here includes two different versions of the part of the solution in which the sizes of the fixed sets are found. The first version uses the permutation method and the second version does not.

We are regarding two coloured tetrahedrons as the same if one can be obtained by rotating the other. So we consider the action of the group $S^+(\text{tet})$ on the set of all possible coloured tetrahedrons in fixed positions. There are 4^4 coloured tetrahedrons in fixed positions.

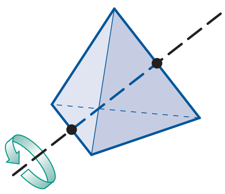
The symmetries in $S^+(\text{tet})$ are of three different geometric types, as follows.

(a) The identity symmetry.

(b) Rotations through $\pm 2\pi/3$ about axes through vertices and centres of opposite faces (four such axes; two such rotations about each).

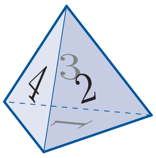


(c) Rotations through π about axes through midpoints of opposite edges (three such axes; one such rotation about each).



Finding the sizes of the fixed sets using the permutation method

We can label the tetrahedron as shown below.



The sizes of the fixed sets for the group action are as given below.

Symmetry g	Example permutation	Number of cycles	$\lvert\text{Fix}\, g\rvert$
identity, (a)	(1)(2)(3)(4)	4	4^4
8 of type (b)	(1 2 3)(4)	2	4^2
3 of type (c)	(1 4)(2 3)	2	4^2

Alternative: finding the sizes of the fixed sets without using the permutation method

We consider the three different geometric types of symmetries in $S^+(\text{tet})$ in turn.

(a) The identity symmetry (type (a)). This fixes all the coloured tetrahedrons, so $\lvert\text{Fix}\, e\rvert = 4^4$.

(b) Eight rotations of type (b). Let g be such a rotation. The coloured tetrahedrons fixed by g are those in which the three faces with a vertex on the axis of rotation have the same colour. The other face can have any colour. Thus $\lvert\text{Fix}\, g\rvert = 4^2$.

(c) Three rotations of type (c). Let g be such a rotation. The coloured tetrahedrons fixed by g are those in which, for each of the two edges intersected by the axis of rotation, the two adjacent faces have the same colour. (The rotation g transposes the four faces in two pairs.) Thus $\lvert\text{Fix}\, g\rvert = 4^2$.

Applying the Counting Theorem

By the Counting Theorem, the number of orbits is

$$\begin{aligned}\tfrac{1}{12}\left(4^4 + 8 \times 4^2 + 3 \times 4^2\right) &= \tfrac{1}{12} \times 4^2\left(4^2 + 11\right)\\ &= \tfrac{4}{3} \times 27\\ &= 4 \times 9\\ &= 36.\end{aligned}$$

Thus there are 36 different coloured tetrahedrons.